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Models and Properties of Financial Limit Order Books

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1 Introduction

Financial contracts are traded all over the world either over-the-counter or through market mechanism as organized exchange. In over-the-counter market, where products are highly tailored and illiquid, transactions usually take place between two well defined parties. No rules for trading exist and each transaction is negotiated separately. In organized exchanges products are standardized and more liquid. Each market (exchange) has it own rules. Two main (market) mechanisms for exchange trading exist, namely auction and continuous trading (continuous double auction). Part of the markets operate as pure limit order markets, while still greatest proportion as mixed markets [Luc01]. Economists refer this as a market microstructure or microscopic mechanism of price formation and evolution.

1.1 Auctions

No trade takes place during the auctions in financial markets. Order book (where all buy and sell offers are stored) is usually kept hidden. After the accumulation of offers the auction is closed and solved regarding maximum trade volume criterion. There are also alternative criteria. Sometimes (such as electricity exchanges[1]) prices are matched according to so called clearing house mechanism, which usually means multiple unit uniform-price auction, or discriminatory (double) auction, see more [Kri02, Chapter 12: An Introduction to Multiple Object Auctions]. Termination time of the auction is typically random [BP03, p. 80]. In stock exchanges there could be auctions during the day (continuous trading is not allowed). For instance in Helsinki Stock Exchange there is morning and evening auctions before and after the period of continuous trade. These are conducted as sealed (when no information on offers are made available for participants) and (semi) open (when at most five best bid and ask offers are shown from order book)[2]. Auctions

[1] www.nordpool.no
[2] www.omx.com

are also well utilized in primary markets (bond and stock issues) all over the world.

1.2 Continuous trading

Continuous trading (also continuous double auction) is the way exchanges work most of their time. In continuous double auction setting no auctioneer is required. Participants send their buy and sell orders and a trade will occur, when predefined conditions are filled. This mechanism is well suited to modern financial markets and its ability to promote price formation (convergence to competitive equilibrium) and efficient allocations are evident [CF96]. In continuous trading framework two basic types of orders exist, limit orders and market orders. The limit order is an order to buy or sell specified amount of product at specified price (or more favorable). Thus new limit order either adds to the order book or generates a trade (transaction price and volume are recorded). There are two prices that can be considered, trade price and mid-price. Trade price is the transaction price of the trade. Several studies [Luc01] [Ros03] have concluded that times between trades are roughly distributed as exponential. Thus trades tend to occur as Poisson process. The mid-price on the other hand is determined as, $m(t) = \frac{1}{2}(\alpha_1(t) + \beta_1(t))$, where $\alpha_1(t)$ is the best ask and $\beta_1(t)$ is the best bid. This quantity can thus be followed in real time as opposed to trade prices, which can only be followed in event time. Orders that are stored in order book are referred as quotes. The market order is an order to sell or buy without price limit, that is, trade takes place at current bid or ask. Limit order market refers to market, where all participants can place limit and market orders. Quote driven market refers to situation, where only market orders are available to final clients [BP03, p. 81]. The bid is the best (maximum) buy price and the ask is the best (minimum) selling price at any given time. The difference between these two at t is so called bid-ask spread, s_t. Transaction cost is thought as $\frac{1}{2}s_t$. Order book is a collection of limit sell- and buy orders at any given time t. Example of an order book is shown in Figure 1 next page. Limit orders have a tendency to decrease the spread, since they fill possible price gaps in order book, whereas market orders will have contrarian effect. Providing liquidity thus equals placing limit orders. Tick refers to the smallest price interval between two prices. Tick's size is typically ten basis points (10^{-3}). For instance, in Helsinki Stock Exchange ticks size is 0.01 EUR for all stocks[3] and in Euro Next 0.01 cent for stocks less than 50 EUR, 0.05 for stocks between 50-100 EUR and 0.10 for stocks more than 100 EUR[4].

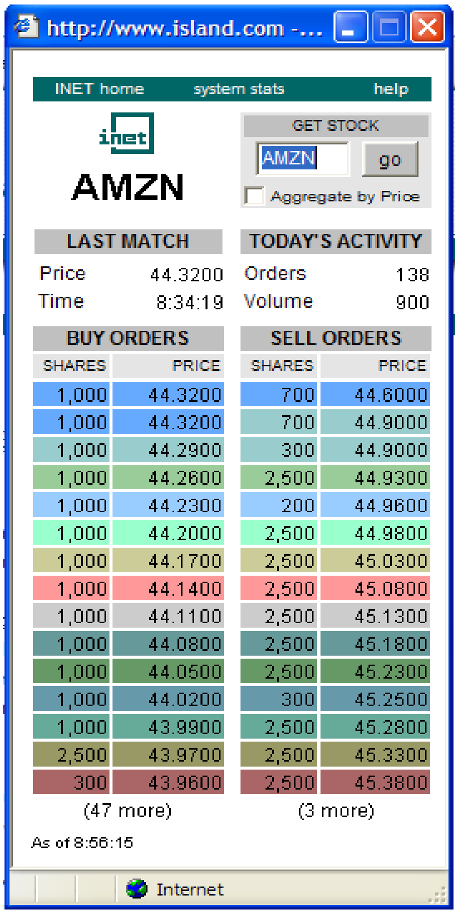

Figure 1: An example of a limit order book for Amazon.com's stock.

There are several other combined order types. Some of these are,

[3]www.hex.com

[4]www.euronext.com

Stop-loss order. This becomes market order as soon as a specific price has been hit. Purpose is to close out position if unfavorable price movements take place.

Stop-limit order. A combination of stop and limit order. This becomes limit order of the specified price (limit price) as soon as other specified price (stop price) has been hit.

Market-if-touched order. This is executed as market order after a trade of specific price or more favorable price has occurred.

Market-not-held order. Market order whose execution is delayed.

Time-of-day order. Specific time period exists when this order can be executed during the day.

Fill-or-kill order. This must be executed immediately or not at all.

These are from [Hul05, Chapter 2: Mechanics of futures markets.]. It should be noted however that only few order types are usually accepted in the particular market [Luc01].

Continuous Double Auction (CDA) can be thought as a multilateral generalization of the process of haggling between seller and buyer. Automated trading means that there are artificial agents that place orders to provide liquidity. Automated trading is fully carried out in Europe and Japan while open outcry system is being slowly replaced by automated trading in U.S. [KO04].

1.3 Motivation for studying order books

Even simple trading mechanism may give raise to price anomalies, such as, fat-tails and volatility clustering, when intraday price statistics are considered. Different assumptions on agents' decision making process might have an impact on stylized facts. Different agents' expectations on price movements may also have an effect on market price. This paper gives a brief review of current order book literature and illustrates simulations of a simple mechanism with agents using random decision making process. It is shown that even the lack of strategies (random decision making) will give raise to fat-tails and volatility clustering. The tail density of individual agents' offer price distributions has little impact on stylized facts in order book level. This notion could be interest of further order book studies.

This paper is organized as follows. Section 1 is an introduction, section 2 introduces some statistical tools for order book analysis and statistical properties of financial limit order markets, section 3 shows some proposed models from current literature, section 4 gives simulation examples of simple mechanism, where order (buy or sell with equal probability) is generated each step from normal and Lévy distributions latest same type of order being the expected value. Section 5 concludes. Appendix A, gives some schemes for generating fat-tailed random variates.

2 Statistical properties

2.1 Definitions and tools for analysis

Buy and sell orders are referred as $\alpha_i(t)$ and $\beta_i(t)$ respectively. At a given time t, prices in order book will satisfy,

$$\cdots \leq \beta_m(t) \leq \cdots \leq \beta_2(t) \leq \beta_1(t) < \alpha_1(t) \leq \alpha_2(t) \leq \ldots \alpha_m(t) \leq \ldots \quad (1)$$

Spread is thus $\alpha_1(t) - \beta_1(t)$. Price gap refers to price differences in order book e.g. the difference between the best price and the k th best price. Liquidity means the markets' ability to absorb new orders. Volatility clustering means, that volatility is not constant in time, i.e., large changes tend to be followed by large changes. Fat-tails, on the other hand, refer to conditions, where tails of the probability distribution (pdf) are heavier than in normally distributed case. Volatility feedback,

$$\sigma(t) = \sigma(\sigma_{\mathcal{I}}, t), \quad (2)$$

that is, volatility depends on the intraday volatility of the traded asset. Anomality of the price process usually means just that increments are not normally distributed. Depth of the order book refers the liquidity in order book. It can be measured as a price impact that a large market order will have [MM01], this is also referred as virtual price impact as opposed to real price impact. In reality, several speculative orders will follow large market order thus smoothing the virtual impact. Other measures for the depth of the order book exist. [FGLSS04] defined it as a density of limit orders in the order book. Shape of the order book is referred as the average number of quotes in queue at the given distance from the best price. This reflects investors' anticipations (willingness to wait, that price goes down significantly).

Hurst Exponent. Hurst exponent denoted as H offers a mean for characterizing temporary fluctuations in time series. Hurst exponent is defined via following relation,

$$H = \frac{\log(R/S)}{\log(T)}, \tag{3}$$

where T is the length of the time serie and R/S is the rescaled range of the time serie. Hurst exponent has a clear and relevant meaning in finance applications. It provides simultaneous measure of long-term correlation and variation. One method to estimate H from the time serie (x_i) is rescaled range algorithm, which works as follows. First following quantities are calculated,

- $E_\ell[x] = \frac{1}{\ell}\sum_{i=1}^{\ell} x_i$
- $X(t,\ell) = \sum_{i=1}^{t}(x_i - E_\ell(x))$
- $R(\ell) = \max_{1\leq t\leq \ell}(X(t,\ell)) - \min_{1\leq t\leq \ell}(X(t,\ell))$
- $S(\ell) = \sqrt{\frac{1}{\ell}\sum_{i=1}^{\ell}(x_i - E_\ell(x))^2}$
- $C\ell^H = E[R(\ell)/S(\ell)]$

Rescaled range is then estimated by calculating the average R/S over multiple regions (size of ℓ) and calculating the slope of regression line in $\log(\ell)$ vs. $\log(R(\ell)/S(\ell))$. This method is straightforward, but unfortunately has poor convergence properties [BS94].

For normally distributed increments in time series $H = \frac{1}{2}$, whereas for persistent random walk (under-diffusive) $H < \frac{1}{2}$ and for anti-persistent random walk (over-diffusive) $H > \frac{1}{2}$.

Variogram. Variogram of stochastic process is defined as,

$$\mathcal{V}(\ell) = E_n[(x_{k+\ell} - x_k)^2] \tag{4}$$

Variogram is an useful tool in characterizing a given process, i.e., checking whether the process is pure random walk, independent random variates or in which time scales it exhibit certain type of behavior. Variogram can be used to observe the near absence of linear correlations, it measures how much on average price differs between two instant of times.

Correlogram. Correlogram or correlation function of the time serie is defined as,

$$\mathcal{C}(\ell) = E_n[x_{k+\ell}x_k] - E_n[x]^2. \tag{5}$$

Sometimes variogram may be better way to characterize the process since using correlogram contains limitations ($E[x]$ must be defined, which is not always the case), see more on these methods from [BP03, Chapter 4: Analysis of empirical data] and [EKM97, Chapter 6: Statistical methods for extremal events]. Volatility is clearly, $\sigma = \mathcal{C}(0)$.

Correlograms and variograms are widely used in geostatistics [BM98] and had been suggested for measures of correlation and variation in financial time series by Bochaud and Potters [BP03]. It should be noted, however, that they can be used for qualitative checking, only. This is because their absolute values are scale dependent indeed. If one desires to use them otherwise, time series must first be scaled to the same range to have comparable measures.

2.2 Generic features

Stock prices and returns have been known to be fat-tailed for 40 years [Fam65]. This is especially the case when intraday price statistics is considered. There has been plenty of discussion whether one should consider absolute or relative price increments in high frequency financial data. Several facts speak for the instability of financial markets; trading techniques evolve, participants change, new products and trading companies are introduced, etc. Thus using long-term time series to long-term analysis is not justified [BP03, p. 88]. Bochaud and Potters [BP03, Chapter 8: Skewness and price-volatility correlations] [BP02] argue that the choice of returns rather than price increments is not justified. It has been found [FGLSS04] that the distribution of large price changes is due gaps in the order book. Farmer et. al. [FGLSS04] conclude that orders do not have price impact themselves, but gaps of certain prices in order book cause large price changes, that is, price fluctuations that are driven by liquidity fluctuations.

Limit order markets are also far from continuous limit. Demand and supply curves at any time t are step-like functions with large jumps and periods of long flat [FGLSS04]. In continuous time, the probability that the closing price of the stock will equal opening price is zero whereas before decimalilization (when stocks were quoted 1/8 th and 1/16 th of dollar) the probability for typical U.S. stock was 20 % [BP03]. Orders have a tendency to cluster both in position and size [CS01]. Volatility feedback is an important feature, which has been found to give raise to fat-tails and volatility clustering [RCFM01] [Rab03], although some studies [RC05] have concluded that intraday return distributions are leptokurtic even without volatility feedback. In double auction mechanism fat-tails of intraday returns are recovered without any particular assumptions about agents' behavior [Mas00].

Raberto and Cincotti [RC05] argue that the normal distribution of daily returns is caused by the fact that limit order books are emptied at the beginning of every trading day. The shape of the order book (distribution of volume from best price) has found to have hump shape [BP03] [Ros03]. Bouchaud and Potters [BP02] also studied a typical life time of a limit order as a function of the distance from the best price. Cancel rate was found to be non-uniform, in other words, life time of a given order increases when the distance from mid-price increases. Bochaud and Potters [BP02] argued that the proposed quantitative theory should reproduce this fact. Challet and Stinchcombe [CS01] found that life times tend to peak at 60, 90, 180, 300 and 600 seconds. Functional form of price impact function varies from market to market and from stock to stock [FL04]. For the shape of the price impact function both power-law [PGGS02] and logarithmic [BP02] forms have been suggested. Analysis of the price impact is presented in [FGLSS04] and [FL04].

Limit and market orders are both found to be asymptotically power-law distributed [ZF02] [WSC02]. The limit orders' sizes seem be [Luc01] good fit to,

$$P(x) = \frac{1}{x} e^{-(A - \ln x)^2 / B}, \tag{6}$$

where A and B vary around 7 and 4, respectively. This has indeed effective power-law exponent of 1. For more studies on effective tail exponents, see [ZF02] [RC05] and [RSM02]. Price process is found [WSC02] to be over-diffusive $H > \frac{1}{2}$ in short-term and have a crossover to random walk (diffusion) $H = \frac{1}{2}$ process in long-term.

It should be noted that, idiosyncrasies illustrated in this subsection, are not important for the superficial understanding, but might be crucial to implementation of quantitative theory.

3 Proposed models

A complete theory of limit order market would provide with statistical description of the order book dynamics and price process. A good model would be simple enough for analytical treatment, but should be able to reproduce some important generic features ('stylized facts') of real markets. As in other modelling, order book models have approached reality progressively, from simple toy models [Mas00] to sophisticated agent-based simulation models [RCFM01], agents endowed with learning and optimization capabilities and ability to utilize (make a profit) from order book abnormalities [LeB02] [Rab03]. Steps for order book modelling include,

1) Making assumptions about traders behavior.
2) Combining effect of traders, that is, the flow of sell and buy orders and their cancelations. These are expressed conveniently as distributions.
3) Sell and buy orders' distributions in order book.
4) Distribution for transaction prices (or mid-price).

Typical assumptions behind the model are,

a.) Zero-intelligence, that is, non-strategic behavior. Decision making process in nearly random.

b.) Homogenic trader population.

c.) Poisson arrival process for orders.

d.) No cancelation possibility (some models).

e.) No herding, that is, traders act independently.

f.) Anonumous traders (orders have ID's).

Some generic features such as fat-tails and volatility clustering are produced by simple assumptions [RC05]. Simplest model is presented by Maslov [Mas00]. Although some generic features are recovered by the toy models; authors in [CS01] found that in continuous-double auction setting the model of three processes (orders, executions, cancellations) is required to produce necessary stylized facts.

Based on their statistical analysis Challet and Stinchcombe [CS01] proposed a model (equivalent particle model). As in all particle models, orders were thought as particles with price as spatial position, size as mass and tick as mesh size in grid. In their model order (particle) X would have to states α and β, that is, sell or buy. There were two possible events, orders can either be deposited or evaporated. These events occur with probabilities p and q. Prices are drawn from Gaussian. This model was able to produce fat-tails and volatility clustering. Short-coming is that this has no over-diffusive behavior. Recovery of over-diffusive behavior of price time series in short term and crossover to random walk one need to include somewhat strategic behavior (e.g. herding) in to the model [BP03, Chapter 20: Simple mechanism for anomalous price statistics] [CS01]. Exclusion particle models, on the other hand, are well known to have the ability to produce this crossover, the cancellation (evaporation) rate being the critical parameter regarding crossover regime, see [WSC02] and references therein.

Luckock [Luc01] proposed a statistical model of a limit order market. Underlying supply and demand functions were modelled as intensity functions of a Poisson processes for incoming limit orders. When these were given stationary solutions to $\alpha_1(t)$ and $\beta_1(t)$ could be derived. The model was build on assumptions a, c, d, e and f. In addition tick size was zero (i.e. continuous model) and underlying supply and demand functions are time-independent. Two important quantities were specified, x_{max} and x_{min}. These have an interpretation as 'support' and 'resistance' levels and will serve as a quantitative explanation of these phenomena. Levels are defined as follows,

$$\begin{aligned} x_{min}(t) &= \inf\{x : B(t,x) < 1\} \\ x_{min}(t) &= \sup\{x : A(t,x) < 1\}, \end{aligned} \tag{7}$$

where $B(t,x)$ is the survivor function of the $\beta_1(t)$, that is, $B(t,x) = P(\beta_1(t) \geq x)$ and $A(t,x) = P(\alpha_1(t) \leq x)$. The region $[x_{min},\ x_{max}]$ is referred as competitive window since no demand or supply exists outside it. This restriction of price fluctuations, is a clear cause of market microstructure.

Another statistical model is presented in [SFGK02] and a tractable continuous time order book model with proved existence of Markov equilibrium of bids and asks (characterized in closed-form in several cases) is presented in [Ros03]. A survey of market microstructure is presented in [Mad00].

4 Simulation examples

This section illustrates simulations of a simple limit order market model. In this setting, buy and sell orders aroused with equal probability. Sizes of the orders were distributed uniformly (from 0 to 10000). Tick size with regard to order sizes was 100, reflecting clustering. Prices were distributed either normally or as Lévy. Tick size with regard to price was 0.01. Figure 2 below shows results, when prices were distributed normally,

$$\alpha(t) = \mathcal{N}(\alpha(t-1), \sigma)$$
$$\beta(t) = \mathcal{N}(\beta(t-1), \sigma), \quad (8)$$

where σ was 5% of starting price 200.

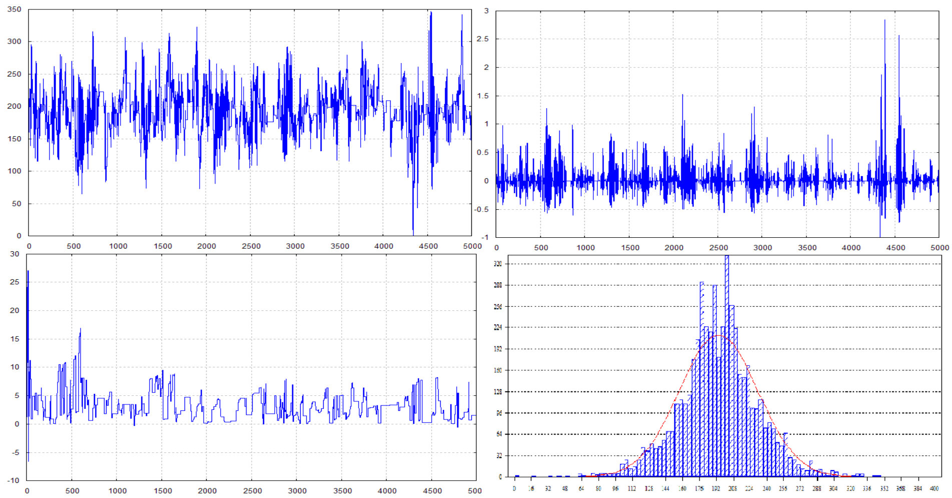

Figure 2: Results of a simulation, when offers were normally distributed around latest offers. (a)Left above: prices, (b) right above: returns, (c) left below: bid-ask spread, (d) right below: histogram of prices.

Some generic features were clearly recovered, namely fat-tails of returns and price increments (although they are nearly normally distributed to the eye) and volatility clustering. Figure 3 shows results when prices were distributed as follows,

$$\alpha(t) = \mathcal{L}(\alpha(t-1), \sigma, \gamma, \delta)$$
$$\beta(t) = \mathcal{L}(\beta(t-1), \sigma, \gamma, \delta), \quad (9)$$

where σ was 1% of starting price 100, $\gamma = 1.89$ and $\delta = 0$.

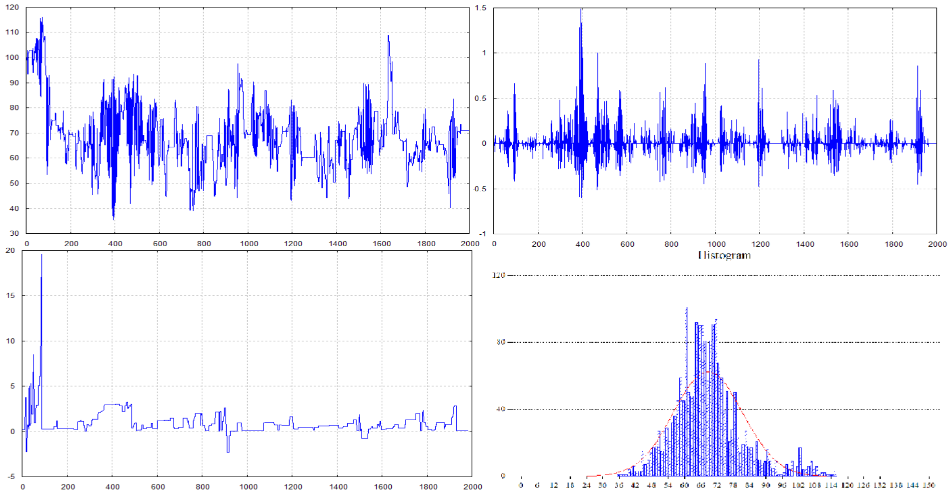

Figure 3: Results of a simulation, when offers are Lévy distributed around latest offers. (a)Left above: prices, (b) right above: returns, (c) left below: bid-ask spread, (d) right below: histogram of prices.

These results have the same statistical properties as for normally distributed prices. Thus the fact that double auction mechanism gives a raise to price anomalities is reflected by the simulations. Q-Q-plots of trade prices are shown in figure 4 below. In both cases, normally and Lévy distributed offer prices, Shapiro-Wilk test rejects normality of trade prices.

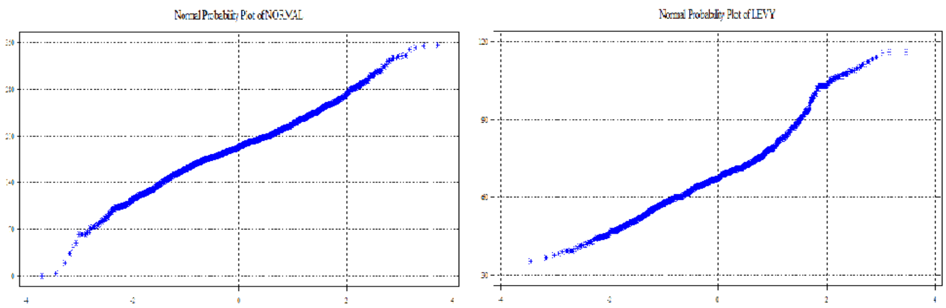

Figure 4: Q-Q-plots of trade prices, a) left, offer prices normally distributed and b) right, offer prices Lévy distributed.

Figure 5 shows variograms of the price statistics both in real time and in trade time i.e., regions with no trades are removed from time series.

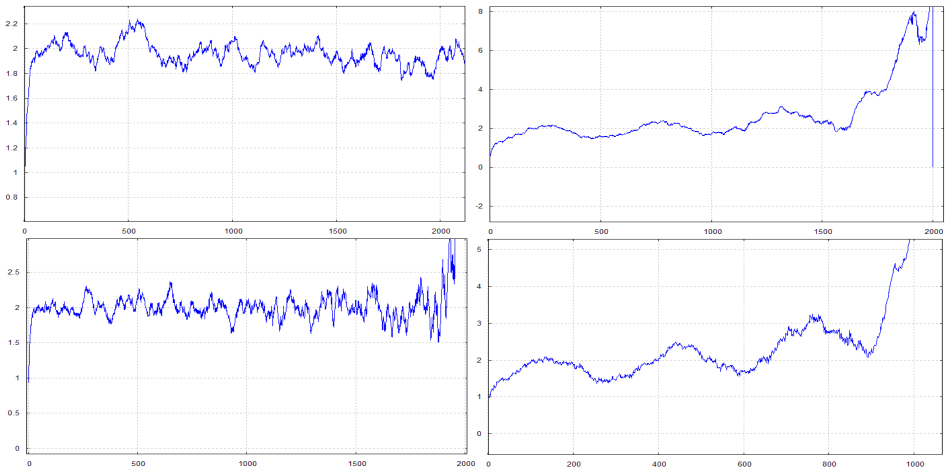

Figure 5: Variogram (a)Left above: normally distributed offers in real time, (b) right above: Lévy, real time, (c) left below: normal, event time, (d) right below: Lévy, event time.

It is shown that removing times of constant trade prices (i.e. ranges when no trades take place) has no impact on the shape of the variograms. For normally distributed offers, the variogram increases rapidly from linear behavior to nearly constant (fluctuating around 2). This corresponds to mean-reverting stochastic process characterized by large mean-reversion speed factor (i.e. process is nearly independent random variates). For Lévy distributed offer prices, there is a trend in the variogram, suggesting that corresponding process is closer to random walk than in normally distributed case. Figure 6 shows autocorrelation functions in the same manner.

Significant autocorrelations exist. These are not affected by the transition from real time to trade time. Autocorrelations with larger time spans seem to be greater.

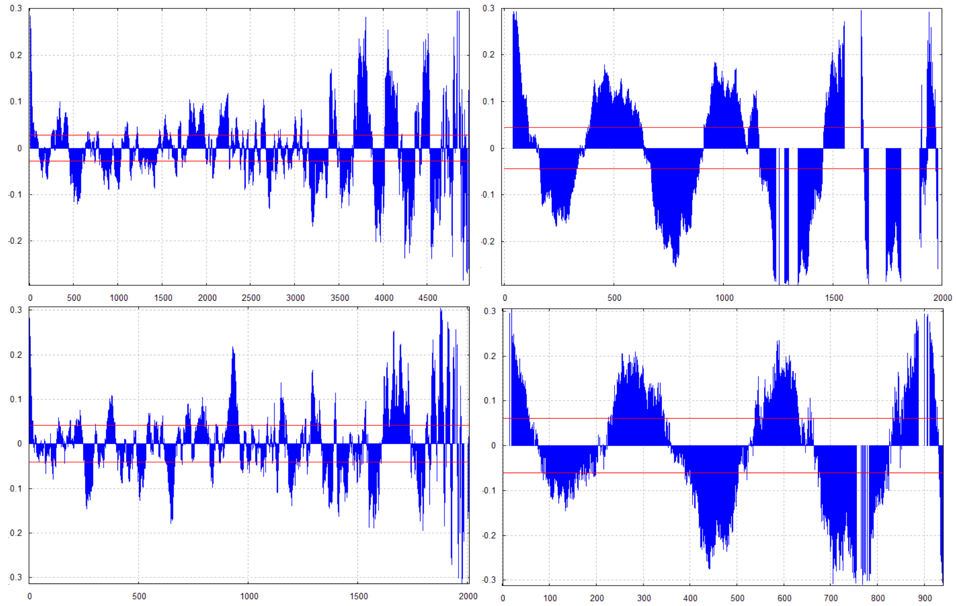

Figure 6: Autocorrelations (a)Left above: normally distributed offers in real time, (b) right above: Lévy, real time, (c) left below: normal, event time, (d) right below: Lévy, event time.

As mentioned earlier, a well know fact in high-frequency finance is long-term correlations in time series, that is, short-term behavior is nearly random, but in the long-term regularities can be observed. The processes are referred to as long-memory processes. In particular, price increments δp have short-term correlations, while $|\delta p|$s or $(\delta p)^2$s have long memory manifested by slow decay in correlation function. It has been successfully fitted to power-law [MM01].

Simulations of this very simple mechanism seem to produce price process that is in line with these facts.

5 Conclusions

Today many financial markets operate as limit order markets in automated exchanges. Every mechanism differs how orders are sent and executed, but in common, the limit order mechanism provides liquidity easily. Modelling this mechanism has several benefits both in practical and scientific interest. Understanding of the mutual interplay of limit-, market orders and cancellations is essential part of modelling.

There are significant number of important features of real markets that are not part of basic models assumptions, however. Participants do use strategies after all. Delays, that is, the actual order book and what one sees on his or her screens have an impact on price statistics indeed [MM01]. Existence of narrow competitive window as suggested in [Luc01] is questionable since market forces (supply and demand) evolve overtime and dominant market players exist after all. Exchanges use incentives to encourage certain type of desirable behavior of the participants, e.g., in Island[5] liquidity provision and patience are rewarded. Even in modern financial markets, perfect competition cannot be taken for granted [BBS03]. Competition is not perfect, but sometimes lack of competition among traders is compensated by the competition among exchanges, however. Financial markets are uncompetitive when only small groups have access to market and relevant information [CR00]. Even if access to information, reduction of trading costs and new technology are important in promoting competition, market power (liquidity supply in particular) has still not been eliminated in financial markets [BBS03]. Thus market perfectness conditions a la Stigler [Sti57] may not hold which had led some authors to work with other set of assumptions [Glo94] [BMR00].

This paper presented simulations of a simple order book model, which recovered some generic features of the real price process.

[5] www.island.com

A Generating fat-tailed random variates

Here u_i is a realization of $U \sim [0,1]$ and Υ is 1 with probability $\frac{1}{2}$ and -1 with probability $\frac{1}{2}$.

A.1 Power-law variates

Density function for power-law (Pareto) distribution reads,

$$P(X = x) = \frac{B}{|x|^{1+\gamma}}, \tag{10}$$

where B is tail amplitude or scale parameter and γ is tail exponent. A varible following this law can be generated as follows,

- Draw, u and Υ
- Calculate $P = \frac{\sqrt{(\gamma-2)(\gamma-1)}}{2\sqrt{2}}\Big(1 + |\Upsilon(u^{-1/\gamma} - 1)|\Big)^{-\gamma-1}$

P is Pareto distrubuted with tail parameter γ and unit variance(provided that exists, i.e. $\gamma > 2$).

A.2 Student variates

Student distribution has power law tails and it's density function with γ degrees of freedom reads,

$$P(x = X) = \frac{1}{\sqrt{\pi}} \frac{\Gamma((1+\gamma)/2)}{\Gamma(\gamma/2)} \frac{a^{\gamma}}{(a^2 + x^2)^{(1+\gamma)/2}}. \tag{11}$$

This converges towards normal distribution, when $\gamma \to \infty$ (square of tail parameter a^2 should be scaled as γ, however) and towards Cauchy distribution, when $\gamma \to 1$. Student distributed random variates with $\gamma = 4$ degrees of freedom can be generated as follows,

- Draw u and Υ
- Set $\tau = 2\cos(\frac{1}{3}\arccos(1 - 2u) + \frac{4\pi}{3})$

- Set $\xi = \frac{\Upsilon\sqrt{2}\tau}{\sqrt{1-\tau^2}}$

- Calculate $P = 3(2 + z^2)^{-\frac{3}{2}}$

P is Student distributed with $\gamma = 4$ degrees of freedom and unit variance.

A.3 Lévy variates

There is no analytical expression for Lévy variate's density function. Characteristic function of density (Fourier transform) reads,

$$\hat{A}_\gamma(z) = e^{-c(\gamma)T|z|^\gamma\left(1+i\delta\tan(\gamma\pi/2)\frac{z}{|z|}\right)}, \tag{12}$$

here $c(\gamma)$ is constant and T is amplitude. The constant is,

$$(\gamma\Gamma(\gamma-1)\frac{\sin(\pi\gamma/2)}{\pi})^{-1} \quad \text{when } 1 < \gamma < 2 \tag{13}$$

and

$$((1-\gamma)\Gamma(\gamma)\frac{sin(\pi\gamma/2)}{\pi\gamma})^{-1} \quad \text{when } \gamma < 1. \tag{14}$$

Parameter $0 < \gamma < 2$ is tail exponent, which determines heaviness of the tails. When $\gamma \to 2$ distribution converges towards Gaussian and when $\gamma = 1$ it reduces to Cauchy distribution. Asymmetry parameter $-1 < \delta < 1$ describes the relative weight of tails. When $\delta = 0$ distribution is fully symmetric and when $|\delta| = 1$ it is fully asymmetric. Lévy distributed variates with tail exponent γ and asymmetry parameter δ can be generated as follows,

- Draw u_1 and u_2 from $U \sim$ [0,1]

- Set $w_1 = \pi(u_1 - \frac{1}{2})$

- Set $w_2 = -\log u_2$

- Set $\Phi = \cos^{-\frac{1}{\gamma}}\left(\arctan\left(\delta\tan\frac{\pi\gamma}{2}\right)\right)$

- Set $\Psi = \frac{\arctan\left(\delta \tan \frac{\pi\gamma}{2}\right)}{1-|1-\gamma|}$

- Calculate $P = \Phi \frac{\sin(\gamma(w_1+\Psi))}{\cos^{\frac{1}{\gamma}}(w_1)} \left(\frac{\cos(w_1-\gamma(w_1+\Psi))}{w_2} \right)^{\frac{1-\gamma}{\gamma}}$

P is Lévy distributed with tail exponent γ, symmetry parameter δ and amplitude such that $T/c = 1$. This method is from [CMS76] and reduces to traditional Box-Muller for generation of normally distributed random variates when $\gamma \to 2$.

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