Preference Programming with Incomplete Ordinal Information

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Abstract: In this paper, we develop the RICHER method (*Rank Inclusion in Criteria Hierarchies with Extended Rankings*) which extends uses of incomplete preference information in value trees by allowing the decision maker (DM) to provide incomplete ordinal preference statements about (i) the relative importance of attributes and (ii) the relative performance of alternatives with regard to a set of attributes. Such statements can be elicited by asking the DM to associate a set of rankings to a set of alternatives (e.g., 'alternatives x^1 and x^2 are among the three best ones with regard to costs') or attributes (e.g., 'the most important attribute is either a_1, a_2 or a_3'). Because statements of this kind may lead to non-convex sets of feasible parameters, we develop equivalent mixed integer linear programming (MILP) formulations which allow such statements to be combined with any preference programming methods that correspond to linear inequalities. The potential of RICHER is illustrated with an example on the siting of an office facility.

Subject classifications: Decision analysis, multiple criteria; Programming, linear, algorithms; Utility/preference, estimation

1 Introduction

Value tree analysis – which has a solid foundation in multi-attribute value theory (MAVT) (Keeney and Raiffa 1976) – is widely employed to address multi-criteria decision making (MCDM) problems across a range of application domains (see, e.g., Corner and Kirkwood 1991, Hämäläinen 2004, Keefer, Kirkwood and Corner 2004). In value tree analysis, the decision problem is approached by (i) structuring the relevant *alternatives* and *attributes*, (ii) assessing the performance of alternatives on the relevant attributes through *scores*, (iii) measuring the relative importance of attributes through *weights*, and (iv) computing for each alternative its *overall value* which serves as an aggregate measure for the development of decision recommendations.

Yet, the elicitation of complete preference information may be problematic due to reasons such as urgency of the decision, lack of resources, or the presence of intangible attributes (see, e.g., Schoemaker and Waid 1982, Weber 1987, Weber and Borcherding 1993, Pöyhönen and Hämäläinen 2001). In consequence, several methods for the modeling of incomplete preference information in value trees have been developed. Many of these allow the decision maker (DM) to specify interval-valued ratio statements about attribute weights or other model parameters (e.g., Arbel 1989, Salo and Hämäläinen 1992, 1995, 2001). From the modeling perspective, such statements correspond to linear constraints so that corresponding value intervals can be obtained from linear programs. If the resulting intervals do not allow the best alternative(s) to be determined, additional insights can be obtained by examining which alternatives outperform others in view of different decision rules (see, e.g., Salo and Hämäläinen 2001).

In a related stream of research, the alleged difficulties of obtaining a complete preference specification have motivated the use of ordinal information in assessing the relative importance of attributes. Here, a potential benefit of ordinal information is that people can relate to it in a more 'natural' way, or process it more reliably than numerical cues (Moshkovich, Mechitov and Olson 2002): indeed, methods which require the DM to supply numerical estimates have been at worst time-consuming, poorly understood by the DMs, and even unstable and misleading in terms of their recommendations (see, e.g., Payne, Bettman and Johnson 1988, 1993, Tversky, Sattath and Slovic 1988; for a comparison of methods, see Barron and Barret 1996). This has spurred the development of methods where the DM provides a rank-ordering of attributes, which is then used to generate numerical weight estimates (e.g., SMARTER; Edwards and Barron 1994).

Building on contributions in these two streams of research, Salo and Punkka (2005) present the RICH method (Rank Inclusion in Criteria Hierarchies) which accommodates incomplete ordinal preference information by allowing the DM to associate several rankings to a given set of attributes: thus, for instance, the DM may specify five attributes which contain the three most important ones. However, RICH cannot be used for evaluating alternatives with regard to attributes in an analogous manner, even though this extension is conceptually straightforward (i.e., 'five alternatives which contain the three best ones'). Nor can RICH be employed in combination with the many methods of preference programming that model incomplete preference information through linear inequalities.

The RICHER method (i.e., RICH with Extended Rankings) presented in this paper eliminates these limitations. Thus, in addition to statements about the relative importance of attributes, the DM can give incomplete ordinal information about how alternatives perform with regard to (i) a single attribute, (ii) a subset of attributes or (iii) holistically with regard to all the attributes. For example, she may state that 'alternative x^4 is among the two most preferred alternatives with regard to cost', or that 'alternatives x^1 and x^3 are among the three most preferred ones with regard to environmental impacts'. Even other statements are possible: the DM may state that 'alternative x^2 is preferred to x^4 with regard to cost', or make a holistic comparison by stating that 'alternative x^1 outperforms x^2 overall'. Taken together, these statements capture preference information which earlier methods have not been able to deal with.

The rest of this paper is organized as follows. Section 2 discusses earlier methods for the modeling of incomplete preference information in value tree analysis. Section 3 formalizes the use of incomplete ordinal preference information about alternatives and presents a mixed integer linear programming formulation which resolves computational challenges that arise from the non-convexity of feasible regions. Section 4 presents an illustrative example.

2 Elicitation of Incomplete Preference Information

In an early contribution to the modeling of incomplete preference information, Arbel (1989) allows the DM to specify interval-valued ratio statements. These statements $w_i/w_j \in [a, b]$ define a convex set of feasible attribute weights. The PAIRS method (Preference Assessment by Imprecise Ratio Statements; Salo and Hämäläinen 1992) processes such interval-valued statements in hierarchical value trees and derives overall value intervals and dominance structure for the alternatives by solving a series of hierarchically structured LP problems, proceeding from the lower to the higher levels of the value tree. In PAIRS, decision recommendations are based on pairwise dominance, which means that alternative x^j is preferred to x^k if and only if the overall value of x^j is greater than or equal to that of x^k for all feasible scores and attribute weights (see also Hazen 1986, Weber 1987). If the preference statements do not allow the best alternative to be identified, the DM is asked to provide further information, whereby the consistency of the preference model is maintained through consistency bounds. Additional support for the specification of interval-valued ratio statements is offered by the preference programming approach of Salo and Hämäläinen (1995) which features an ambiguity index for measuring the completeness of the preference model.

Salo (1995) considers groups of DMs who provide incomplete preference information about attribute weights and alternatives' scores. His method also admits incomplete information about the relative importance of DMs, as captured by their weights in an additive model of the group's preferences. Again, all preference statements correspond to linear constraints so that value intervals and dominance relations can be obtained through linear programming. The approach has found uses in traffic planning, for instance (Hämäläinen and Pöyhönen 1996).

In their taxonomy of incomplete preference information, Park and Kim (1997) distinguish between 1) weak ranking $w_i \geq w_j$, 2) strict ranking $w_i - w_j \geq \alpha_i$, 3) ranking with multiples $w_i \geq \alpha_i w_j$ 4) interval form $\alpha_i \leq w_i \leq \alpha_i + \epsilon_i$, and 5) ranking of differences $w_i - w_j \geq w_k - w_l, j \neq k \neq l$ ($\alpha_i, \epsilon_i \geq 0 \; \forall \; i$; see also Kim and Ahn 1999). Mármol, Puerto and Fernández (1998, see also 2002) consider computational aspects of such statements and develop an algorithm for computing the extreme points of feasible regions that are defined through weight intervals or, more generally, linear inequalities whose matrix representation is a Q-operator (see also Carrizosa et al. 1995).

The PRIME method (Preference Ratios in Multi-Attribute Evaluation; Salo and Hämäläinen 2001) allows the DM to provide different kinds of incomplete preference information, including holistic comparisons between alternatives, ordinal strength of preference judgements, and ratios of value differences. Full support for PRIME is provided by *PRIME Decisions*©, a decision support tool that offers visual presentations of value intervals and dominance structure. This tool is available free-of-charge for academic use at `http://www.decisionarium.hut.fi/` (see Hämäläinen 2003). It has found uses in the valuation of a high-technology firm, for example (Gustafsson, Salo and Gustafsson 2001).

VIP Analysis© (Dias and Climaco 2000) is another decision support software that allows the DM to use incomplete preference information about the attribute weights in form of linear inequalities. Building on this earlier work, Dias and Climaco (2005) present an architecture for a group version of VIP Analysis. Dias et al. (2002) and Mousseau et al. (2003) accommodate incomplete preference information in the ELECTRE TRI method which makes use of concordance and discordance indexes. Specifically, Dias et al. (2002) consider the development of robust decision recommendations that are supported by all combinations of acceptable parameter values, while Mousseau et al. (2003) describe procedures for eliminating inconsistent constraints that are in conflict with previously entered preference statements.

In a recent paper, Salo and Punkka (2005) develop the RICH method (Rank Inclusion in Criteria Hierarchies) which – unlike earlier rank-based methods (e.g., SMARTER; Edwards and Barron 1994) – allows the DM to specify incomplete ordinal information about the relative importance of attributes through paired sets of attributes and rankings. For example, if there are four attributes, the DM can state that 'attributes a_1 and a_2 are among the three most important ones'. Several rank-orderings are compatible with this statement, for instance a rank-ordering where a_2 is the most important attribute, a_3 is second most important, followed by a_1 and finally a_4 as the least important attribute. The union of these compatible rank-orderings may constitute a non-convex region of attribute weights: Figure 1, for example, shows the feasible region when either a_1 or a_2 is the most important among three attributes.

Insert Figure 1 about here

The RICH method is supported by *RICH Decisions*©, a web-based decision aiding tool at `http://www.rich.hut.fi/` (Salo, Punkka and Liesiö 2003; see also Hämäläinen 2003). To-date, this method has been employed in two reported applications. Ojanen, Makkonen and Salo (2004) consider the use of RICH in the comparison of alternative combinations of risk management methods at an energy utility. Salo and Liesiö (2004), on the other hand, report an application where RICH was employed to support the setting of priorities for a Scandinavian research programme in wood material science.

If incomplete preference information does not lead to conclusive dominance results, it is instructive to examine which alternatives are recommended by decision rules. Such rules can be based on the selection of a representative vector from the feasible parameter set (e.g., central weights; Salo and Hämäläinen 2001) or rank-based weighting methods (e.g., rank order centroid, rank sum, rank reciprocal; see Barron and Barret 1996). Further rules can be derived from the ranges of values that alternatives may attain (Salo and Hämäläinen 2001): this is the case, for instance, in the *maximax* rule (which supports the alternative with the highest overall value), *maximin* rule (which recommends the alternative whose smallest possible value exceeds that of others) and *central values* rule (which recommends the alternative for which the average of its maximum and minimum values is the highest). The *minimax regret* rule, too, is analogous to these, in that it recommends the alternative for which the greatest possible loss of value relative to any other alternative is smallest.

3 Modeling Incomplete Ordinal Information

3.1 Additive Value Function

In MAVT, the decision problem is characterized by attributes and alternatives, denoted by $A = \{a_1, \dots, a_n\}$ and $X = \{x^1, \dots, x^m\}$, respectively (see, e.g., Keeney and Raiffa 1976). The DM's preferences are modeled through a preference relation $\succeq$ such that $x^i = (x^i_1, \dots, x^i_n) \succeq x^j = (x^j_1, \dots, x^j_n)$ if and only if alternative x^i is either more preferred than or equally preferred to x^j. If the relation satisfies the required conditions (such as mutual preferential independence; see, e.g., Fishburn 1970, Keeney and

Raiffa 1976), there is an additive value function $v(\cdot)$ such that

$$x^i \succeq x^j \iff \sum_{k=1}^{n} v_k(x_k^i) \geq \sum_{k=1}^{n} v_k(x_k^j),$$

where $v_k(x_k^i)$ is the value of alternative x^i with regard to attribute a_k (i.e., *score*) and

$$V(x^i) = \sum_{k=1}^{n} v_k(x_k^i) \tag{1}$$

is the *overall value* of alternative x^i.

The most and least preferred achievement levels with regard to the k-th attribute are denoted by x_k^* and x_k°, respectively; these need not correspond to available alternatives. Because the value function is unique up to positive affine transformations, we may assume that $v_k(x_k^\circ) = 0, k = 1, \ldots, n$ and $V(x^*) = \sum_{k=1}^{n} v_k(x_k^*) = 1$ where $x^* = (x_1^*, \ldots, x_n^*)$. We also assume that all attributes are relevant, in the sense that the achievement level x_k^* is strictly preferred to x_k°, i.e., $v_k(x_k^\circ) < v_k(x_k^*)$.

The scores of the additive value function are recorded in a matrix

$$S = \begin{pmatrix} v(x^1) \\ \vdots \\ v(x^m) \end{pmatrix} = \begin{pmatrix} v_1(x_1^1) & \cdots & v_n(x_n^1) \\ \vdots & \ddots & \vdots \\ v_1(x_1^m) & \cdots & v_n(x_n^m) \end{pmatrix},$$

where the j-th row $v(x^j) = \Big(v_1(x_1^j), v_2(x_2^j), \ldots, v_n(x_n^j)\Big)$ contains the scores of alternative x^j, and the i-th column contains the scores with regard to attribute a_i. The normalization conditions $V(x^\circ) = 0$ and $V(x^*) = 1$ are satisfied by scores $s_{ij} = v_j(x_j^i)$ that belong to the set

$$S_0 = \{s \in \mathbb{R}^{m \times n} \mid 0 = s_{\circ j} \leq s_{ij} \leq s_{*j}, i = 1, \ldots, m, j = 1, \ldots, n, \sum_{j=1}^{n} s_{*j} = 1\},$$

where $s_{\circ j} = v_j(x_j^\circ)$ and $s_{*j} = v_j(x_j^*)$. From the score matrix S, the alternatives' overall values are readily obtained as row sums. Also, if attention is restricted to a subset of attributes $A' \subseteq A$, the corresponding value is $v_{A'}(x^i) = \sum_{a_j \in A'} v_j(x_j^i) = \sum_{a_j \in A'} s_{ij}$.

The overall value of alternative x^i in (1) is often written as

$$V(x^i) = \sum_{k=1}^{n} v_k(x_k^i) = \sum_{k=1}^{n} [v_k(x_k^*) - v_k(x_k^\circ)] \frac{v_k(x_k^i)}{[v_k(x_k^*) - v_k(x_k^\circ)]} = \sum_{k=1}^{n} w_k v_k^N(x_k^i), \tag{2}$$

where the difference $w_k = v_k(x_k^*) - v_k(x_k^\circ)$ is the weight of the k-th attribute and $v_k^N(x_k^i) = [v_k(x_k^i)]/[v_k(x_k^*) - v_k(x_k^\circ)] \in [0, 1]$ is the normalized score with regard to this attribute. The multiplication of these two terms in (2), however, leads to non-linearities and associated computational challenges in the analysis of incomplete preference information. We therefore work with the additive representation in (1), in recognition that the corresponding normalized parameters can be obtained from (2), if needed.

3.2 Ordinal Information and Pairwise Bounds

Ordinal preference information about alternatives consists of comparative statements about which alternatives are preferred to others, but which do not convey how much one alternative is possibly preferred to another. Examples of such statements include holistic comparisons between pairs of alternatives with regard to the whole set of relevant attributes (see, e.g., Salo 1995). Such comparisons are elicited, for instance, in conjoint analysis with the aim of informing product design decisions (see, e.g., Pekelman and Sen 1974; Horsky and Rao 1984).

Ordinal preference information about pairs of alternatives can be captured with pairwise bounds:

Definition 1 *Assume that $x^j, x^k \in X$, S is a non-empty subset of S_0 and $A' \subseteq A$ is a non-empty subset of attributes. The pairwise bound $\mu_{A'}(x^j, x^k)$ is*

$$\mu_{A'}(x^j, x^k) = \min_{s \in S} \sum_{a_i \in A'} [s_{ji} - s_{ki}] .$$

If the DM states that x^j is preferred to x^k with regard to the attributes in A', the constraint $\mu_{A'}(x^j, x^k) \geq 0$ must hold (otherwise, there would exist a combination of feasible scores such that $V(x^k) > V(x^j)$, contrary to the DM's preference statement). Conversely, if $\mu_{A'}(x^j, x^k) \geq 0$, then alternative x^j is the preferred one, because its value is greater than or equal to that of x^k for all feasible scores. Thus, pairwise bounds can be used both in preference modeling and determination of dominance results.

Definition 2 *Assume that $x^j, x^k \in X$ and S is a non-empty subset of S_0. Then alternative x^j dominates x^k with regard to the attribute set A' in the sense of (pairwise) dominance iff $\mu_{A'}(x^j, x^k) \geq 0$.*

Even though negative pairwise bounds do not support dominance conclusions, they are helpful because they indicate by how much more the value of some alternative might exceed that of another, subject to the incomplete preference specification. The expression $\max_{x^k \neq x^j} [-\mu_A(x^j, x^k)]$, for example, is the maximum loss of value that could result if the DM were to choose x^j and the alternatives' values are based on feasible scores such that the difference between $V(x^j)$ and some other alternative with a higher value is maximized.

3.3 Incomplete Ordinal Preference Information

Ordinal information can be captured through rank-orderings which associate a ranking with each alternative. Formally, if $X' \subseteq X$ is a subset of alternatives, a rank-ordering is a bijection $r(\cdot; X') : X' \mapsto$

$\{1,\ldots,m'\}$ (where $m' = |X'|$ is the cardinality of X'). Within X', the alternative with the ranking 1 is the most preferred one, followed by the alternative whose ranking is 2, and so on, until the least preferred alternative with ranking m' is reached.

Rank-orderings are denoted by the convention $r(X') = (r_1,\ldots,r_{m'})$ where $r_k = r(x^j;X')$ is the ranking of the alternative with the k-th smallest index in X' (i.e., $\left|\{x^i \in X' \mid i < j\}\right| = k-1$). Because ordinal comparisons between alternatives can be made with regard to a single attribute, several attributes, or the set of all attributes (holistic comparison), it is necessary to specify the set of attributes with regard to which the rank-ordering is specified. In what follows, $r_{A'}(X')$ denotes a complete rank-ordering, specified on $X' \subseteq X$ with regard to the attribute set $A' \subseteq A$.

Following Salo and Punkka (2005), incomplete ordinal information is modeled through statements which associate subsets of alternatives (denoted by I) with subsets of corresponding rankings (denoted by J). For example, if the DM states that alternatives x^1 and x^3 are the two most preferred ones among four alternatives $X' = \{x^1,x^2,x^3,x^4\} \subset X$ when considering attributes $A' = \{a_1,a_2\}$, these sets are $I_{A',X'} = \{x^1,x^3\}$ and $J_{A',X'} = \{1,2\}$. This statement requires that the rankings of alternatives x^1 and x^3 are either 1 or 2, without specifying which one has the ranking one. It is compatible with four rank-orderings $r_{A'}(X') = (r_{A'}(x^1;X'), r_{A'}(x^2;X'), r_{A'}(x^3;X'), r_{A'}(x^4;X'))$, i.e., $(1,3,2,4), (1,4,2,3), (2,3,1,4)$ and $(2,4,1,3)$.

The subsets of alternatives $I = I_{A',X'}$ and associated rankings $J = J_{A',X'}$ need not be of equal size. If there are fewer alternatives than rankings ($|I| < |J|$), we require that all alternatives in I have rankings that belong to J; and if there are fewer rankings than alternatives ($|I| > |J|$), we require that all the rankings in J are attained by alternatives in I. Rank-orderings which fulfill these conditions are *compatible* with I and J. In general, the set of compatible rank-orderings is defined as follows (Salo and Punkka 2005).

Definition 3 *Let $I \subset X' \subseteq X$ and $J \subset \{1,\ldots,m'\}$ where $m' = |X'|$ and $|I|,|J| > 0$. The set of compatible rank-orderings is*

$$R_{X'}(I,J) = \begin{cases} \{r \in R(X') \mid r^{-1}(j) \in I \,\forall\, j \in J\}, & \text{if} |I| \geq |J| \\ \{r \in R(X') \mid \ r(x^k) \in J \ \forall\ x^k \in I\}, & \text{if } |I| < |J|, \end{cases} \tag{3}$$

where $R(X')$ is the set of all rank-orderings defined on X'.

Compatible rank-orderings have several useful properties, as stated in Lemma 1. For instance, rank-orderings that are compatible with I and J are also compatible with their complement sets $\overline{I}$ and $\overline{J}$; and several comparative inclusion relationships also hold (Salo and Punkka 2005).

Lemma 1 *Let $I \subset X' \subseteq X$, $J \subset \{1, \ldots, m'\}$ with $m' = |X'|$ and $|I|, |J| > 0$. Then the following properties hold:*

a) $R(I, J) = R(\overline{I}, \overline{J})$.

b) $|R(I,J)| = \begin{cases} \frac{|I|!(m'-|J|)!}{(|I|-|J|)!}, & \text{if } |I| \geq |J| \\ \frac{|J|!(m'-|I|)!}{(|J|-|I|)!}, & \text{if } |I| < |J| \end{cases}$.

Furthermore, the following relationships hold whenever $I, I_1, I_2 \subset X' \subseteq X$, $J, J_1, J_2 \subset \{1, \ldots, m'\}$:

c) *If* $|I_1|, |I_2| \geq |J|$, *then* $I_2 \subset I_1 \iff R(I_2, J) \subset R(I_1, J)$.

d) *If* $|I_1|, |I_2| \leq |J|$, *then* $I_1 \subset I_2 \iff R(I_2, J) \subset R(I_1, J)$.

e) *If* $|J_1|, |J_2| \leq |I|$, *then* $J_1 \subset J_2 \iff R(I, J_2) \subset R(I, J_1)$.

f) *If* $|J_1|, |J_2| \geq |I|$, *then* $J_2 \subset J_1 \iff R(I, J_2) \subset R(I, J_1)$.

3.4 Feasible Regions

A complete rank-ordering $r_{A'}(X')$ on the set X' implies the following linear constraints on feasible scores

$$S(r_{A'}(X')) = \{s \in S_0 \mid v_{A'}(x^i) \geq v_{A'}(x^j) \text{ if } r_{A'}(x^i; X') < r_{A'}(x^j; X'),\ x^i, x^j \in X'\}. \tag{4}$$

Thus, if $r_{A'}(x^i; X') < r_{A'}(x^j; X')$, then for any $s \in S(r_{A'}(X'))$ the inequality $v_{A'}(x^i) = \sum_{a_k \in A'} s_{ik} \geq \sum_{a_k \in A'} s_{jk} = v_{A'}(x^j)$ ensures that pairwise bound $\mu_{A'}(x^i, x^j)$ is non-negative. Clearly, this rank-ordering does not constrain the scores of alternatives that are not in X'.

For a given set of alternatives $I_{A',X'} \subset X'$ and associated rankings $J_{A',X'} \subset \{1, \ldots, m'\}$, the corresponding feasible score set can be defined as

$$S(I_{A',X'}, J_{A',X'}) = \bigcup_{r \in R(I_{A',X'}, J_{A',X'})} S(r), \tag{5}$$

where $R(I_{A',X'}, J_{A',X'})$ is the set of complete rank-orderings that are compatible with $I = I_{A',X'}, J = J_{A',X'}$, in accordance with (3).

Feasible sets based on incomplete ordinal information may be non-convex, as exemplified by Figure 1. In the RICH method, the resulting computational challenges are largely resolved by noting that the feasible weight region can be divided into subsets $S(r)$ which correspond to a complete rank-ordering r of attributes. Because the extreme points of such subsets can be determined by enumeration, all computations – such as the derivation of dominance structure – can be carried out by inspecting these extreme points.

In contrast, incomplete ordinal preference information about alternatives entails greater challenges, because (i) the consideration of different attribute sets A' in (4) involve different subsets (columns) of the score matrix S and (ii) the alternatives' scores cannot be normalized in the same way as attribute weights. This makes it impossible to enumerate the extreme points of the feasible set (5) in advance. We therefore wish to develop a formulation (i) which captures incomplete ordinal preference information and (ii) which can be combined with preference information that is expressed by linear inequalities.

3.5 Mixed Integer Linear Programming Model

In what follows, we develop a mixed integer linear programming (MILP) formulation for the feasible set $S(I,J) = S(I_{A',X'}, J_{A',X'})$, defined through the specification of sets $I = I_{A',X'}, J = J_{A',X'}$ and corresponding constraints in (4). For notational brevity, we omit references to the set A' in most of this section, in the understanding that $v(x^i)$ refers to the score sum $v_{A'}(x^i) = \sum_{a_j \in A'} v_j(x^i_j) = \sum_{a_j \in A'} s_{ij}$. We also assume that the set of alternatives I contains at least as many elements as the set of rankings J (i.e., $|I| \geq |J|$), which means that all the rankings in J are attained by alternatives that belong to I; this restriction will be removed later on.

The key to our formulation is that (i) the values of alternatives whose ranking exceeds j can be bounded from above by a corresponding 'milestone' variable z_j and (ii) there are at most k alternatives whose value is either higher than or equal to z_j. Specifically, we introduce constraints

$$z_j \leq v(x^i) + (1 - y_j(x^i))M \tag{6}$$

$$v(x^i) \leq z_j + y_j(x^i)M \tag{7}$$

$$\sum_{x^i \in X'} y_j(x^i) = j \tag{8}$$

which must hold for all $x^i \in X'$ and $j = 1, \ldots, m' - 1$, subject to the requirement that $z_j \in [0,1], y_j(x^i) \in \{0,1\}$ (with $M \gg 0$ a large constant). For example, for $j = 1$, the last inequality implies that there is only one $x' \in X'$ such that $y_1(x') = 1$, which means that $y_1(x'') = 0$ for any $x'' \neq x'$. Inserting these into (6) and (7) gives $v(x') \geq z_1 \geq v(x'')$, $\forall\, x'' \neq x'$. In Lemma 2, this result is generalized to show that the values $v(x^i), x^i \in X'$, can be separated by milestone variables $z_1, \ldots, z_{m'-1}$. All proofs are in the Appendix.

Lemma 2 *Assume that $v(x^i) \in [0,1], x^i \in X'$ and let $v^{(1)}, \ldots, v^{(m')}$ be a rearrangement of the values $v(x^i), x^i \in X'$ such that $v^{(i)} \geq v^{(i+1)}, i = 1, \ldots, m' - 1$. Then the constraints (6)–(8) have a feasible* *solution $z_j, y_j(x^i)$. Moreover, any such solution satisfies the constraints*

$$z_j \geq v^{(j+1)} \geq z_{j+1}, j = 0, \ldots, m' - 1$$

where $z_0 = 1$ and $z_{m'} = 0$.

Because $|I| \geq |J|$, every ranking in J is attained by an alternative in I. In view of constraint (8), this means that for any $j \in J$, the number of alternatives $x^i \in I$ for which $y_j(x^i) = 1$ holds, is one larger than the number of alternatives for which $y_{j-1}(x^i) = 1$ holds. We thus have the constraint

$$\sum_{x^i \in I} [y_j(x^i) - y_{j-1}(x^i)] = 1, \qquad \forall\, j \in J \tag{9}$$

where notational conventions $y_0(x^i) = 0$, $y_{m'}(x^i) = 1$ and requirements $z_j \in [0, 1], y_j(x^i) \in \{0, 1\}$ apply. We are now ready to state our first main result for characterizing the feasible region $S(I, J)$.

Theorem 1 *Assume that $I \subset X' \subseteq X$ and $J \subset \{1, \ldots, m'\}$ and $|I| \geq |J| > 0$. Let $S^1(I, J)$ be the set of scores $s \in S_0$ such that constraints (6)–(9) hold for $v_{A'}(x^i) = \sum_{j \in A'} s_{ij}$ and some $z_j \in [0, 1], y_j(x^i) \in \{0, 1\}, \forall\, x^i \in X', j = 1, \ldots, m' - 1$. Then $S(I, J) = S^1(I, J)$.*

The significance of Theorem 1 is that it specifies an MILP formulation which captures incomplete ordinal preference information, as provided through the sets I and J when the DM considers alternatives X' with regard to the attribute set A'. A major advantage of this formulation is that it can be readily combined with other forms of preference statements which correspond to linear constrains on the scores $s_{ij} = v_j(x^i_j)$ (or even on attribute weights $w_j = s_{*j} - s_{\circ j}$).

Yet, a source of concern with the above characterization is that the number of variables and constraints can be large. To reduce the number of the binary variables and the size of the constraint system, we treat the set of rankings J in terms of its sequential subsets, defined as follows:

Definition 4 *Let $J \subset \{1, 2, \ldots, m\}$. Set J is sequential when $a, b \in J, a \leq b$ implies that $k \in J$ for any integer k such that $a \leq k \leq b$. Otherwise, J is non-sequential.*

Thus sets $J = \{1, 2, 3\}$ and $J = \{5, 6\}$ are sequential but $J = \{1, 3\}$ is not, because $2 \notin J$. If the set of rankings is sequential (i.e., there exist j^-, j^+ (with $j^- \leq j^+$) such that $J = \{j^-, \ldots, j^+\}$), the set of feasible scores that are associated with I and J is characterized by the constraints in Theorem 2.

Theorem 2 *Let* $I \subset X' \subseteq X, J = \{j^-, \ldots, j^+\} \subset \{1, \ldots, m'\}$ *and assume that* $|I| \geq |J| > 0$. *Let* $S^2(I, J)$ *consist of scores* $s \in S_0$ *such that constraints (6)–(8) hold for* $v_{A'}(x^i) = \sum_{j \in A'} s_{ij}$ *and some* $z_j \in [0, 1], y_j(x^i) \in \{0, 1\},\ \forall\ x^i \in X', j \in \{j^- - 1, j^+\} \cap \{1, \ldots, m' - 1\}$ *so that the constraint*

$$\sum_{x^i \in I} \left[y_{j^+}(x^i) - y_{j^- - 1}(x^i) \right] = |J| \tag{10}$$

also holds (with notational conventions $y_0(x^i) = 0, y_{m'}(x^i) = 1$*). Then* $S(I, J) = S^2(I, J)$.

A major difference between Theorems 1 and 2 is that the latter involves far fewer variables and constraints. When there are $m' = |X'|$ alternatives and the set of rankings $J \subset \{1, \ldots, m'\}$ is sequential, Theorem 2 involves at most $m' \times 2$ constraints of both (6) and (7), at most 2 constraints (8) and a single additional constraint (10). In contrast, Theorem 1 involves $(m' - 1) \times m'$ constraints of both (6) and (7), $m' - 1$ constraints (8) and $|J|$ additional constraints (9). In addition, the number of binary variables $y_j(x^i)$ is at most $m' \times 2$ in Theorem 2, while Theorem 1 involves $(m' - 1) \times m'$ such variables. From a computational perspective, Theorem 2 thus offers a superior characterization, if its conditions hold.

If the set of rankings J is not sequential, it can still be written as the union of sequential subsets $J = \bigcup_{i=1}^{k} J_i$. In particular, for any $J_i \subseteq J$, we say that J_i is a *maximal sequential subset* of J if J_i is sequential and not a proper subset of any sequential subset of J. Such maximal sequential subsets define a unique partition of J. By Lemma 3 and Definition 5, the case where the DM specifies a non-sequential set J can thus be dealt with by dividing it into these maximal sequential subsets J_i and by imposing the constraints that correspond to the pairs $(I, J_i), i = 1, \ldots, k$.

Lemma 3 *Let* $I \subset X' \subseteq X, J \subset \{1, \ldots, m'\}, |I| \geq |J| > 0$, *and assume that* $J = \bigcup_{j=1}^{k} J_j$ *for some* $J_j, j = 1, \ldots, k$. *Then*

$$R(I, J) = \bigcap_{j=1}^{k} R(I, J_j).$$

We next consider the general case where the DM specifies a sequence of paired sets $(I_1, J_1), \ldots, (I_k, J_k)$. First, if the conditions of Theorem 2 hold for $(I_l, J_l), 1 \leq l \leq k$, the paired statement (I_l, J_l) can be modeled by this theorem. Second, if $|I_l| \geq |J_l|$ but J_l is not sequential, then J_l can be partitioned into its maximal sequential subsets so that the conditions of Theorem 2 hold for I_l and the resulting sequential subsets of J_l. Third, if $|I| < |J|$, the result $R(I, J) = R(\overline{I}, \overline{J})$ from Lemma 1 makes it possible to construct the feasible region $S(I, J) = S(\overline{I}, \overline{J})$ on the basis of the complement sets $\overline{I} = \{x^i \in X' \mid x^i \notin I\}, \overline{J} = \{j \in \{1, \ldots, m'\} \mid j \notin J\}$ for which the inequality $|\overline{I}| \geq |\overline{J}|$ holds. Also, if $\overline{J}$ is not sequential, it can be partitioned into its maximal sequential subsets, whereafter Theorem 2 can be applied to paired combinations of $\overline{I}$ and resulting sequential subsets of $\overline{J}$.

As a result, the DM's statements can thus be modeled through a series of paired sets $(I_1, J_1), \ldots, (I_k, J_k)$ where $|I_l| \geq |J_l|$ and each J_l is sequential. Because feasible scores must satisfy all the corresponding constraints, it follows that they belong to the set

$$S(I_1, \ldots, I_k; J_1, \ldots, J_k) = \bigcap_{l=1}^{k} S(I_l, J_l). \tag{11}$$

This joint region of feasible scores can be characterized through linear constraints, as stated in Theorem 3.

Theorem 3 *Assume that $I_1, \ldots, I_k \subset X'$ are subsets of alternatives and $J_1, \ldots, J_k \in \{1, \ldots, m'\}$ are sequential subsets of rankings such that $|I_l| \geq |J_l| > 0, l = 1, \ldots, k$. Let $S^{\cap}$ consist of those scores $s \in S_0$ for which the constraints (6)–(8) hold for $v_{A'}(x^i) = \sum_{j \in A'} s_{ij}$ and some $z_j \in [0,1], y_j(x^i) \in \{0,1\}$, $\forall\ x^i \in X', j \in \cup_{l=1}^{k} \{\min J_l - 1, \max J_l\} \cap \{1, \ldots, m'-1\}$ so that (10) holds for $J_l, l = 1, \ldots, k$. Then $S(I_1, \ldots, I_k; J_1, \ldots, J_k) = S^{\cap}$.*

The joint feasible set $S(I_1, \ldots, I_k; J_1, \ldots, J_k)$ is never empty, no matter what the sets $I_1, \ldots, I_k, J_1, \ldots, J_k$. This is because for example the value vector $v_{A'}(x^i) = 1/m'$, $x^i \in X'$ satisfies the constraints that are implied by any complete rank-ordering of alternatives in X'.

For any scores in $S^{\circ}_{A'} = \{s \in S_0 \mid x^k \neq x^l \Rightarrow v_{A'}(x^k) \neq v_{A'}(x^l)\}$, the binary variables $y_j(x^i)$ in Theorems 1 and 2 are unique. For if this were not the case, there would exist two different sets of binary variables, say $y_j(\cdot), y'_j(\cdot)$, such that the constraints (6)–(8) are satisfied for $s \in S^{\circ}_{A'}$. Also, in view of constraint (8), there would exist distinct alternatives x^k, x^l such that $y_j(x^k) = 0, y_j(x^l) = 1, y'_j(x^k) = 1, y'_j(x^l) = 0$. But inserting these parameters into constraints (6)–(7) would imply $v_{A'}(x^k) = \sum_{a_j \in A'} s_{kj} = \sum_{a_j \in A'} s_{lj} = v_{A'}(x^l)$, contradicting the assumption that $s \in S^{\circ}_{A'}$.

In view of the above, the binary variables $y_j(x^i)$ can be interpreted so that $y_j(x^i) = 1$ implies that the ranking of x^i is less than or equal to j. From this interpretation it follows that if $y_j(x^i) = 1$, then $y_{j+1}(x^i) = 1$ should hold, too. Building on the procedure in the proof of Theorem 1, we can therefore add monotonicity constraints

$$y_{j_1}(x^i) \leq y_{j_2}(x^i), \qquad \forall\ x^i \in X', \tag{12}$$

where $j_1 \in J_{ind} = \cup_{l=1}^{k} \{\min J_l - 1, \max J_l\} \cap \{1, \ldots, m'-1\}, j_1 \neq \max J_{ind}$ and j_2 is the least other ranking in J_{ind} such that $j_2 > j_1$. The resulting constraints do not eliminate any scores, but may contribute to better computational performance. In the example of Section 4, for instance, the computation of pairwise bounds and alternatives' value intervals declined from 65 seconds to 14 seconds due to the addition of monotonicity constraints.

The feasible regions in Theorems 2 and 3 can also be used to model incomplete ordinal information about the relative importance of attributes, using statements that are elicited in the RICH method (e.g., 'cost is among the three most important attributes'). Specifically, let x^{i*} denote a consequence such that $x_j^{i*} = x_j^*$ if $j = i$ and $x_j^{i*} = x_j^\circ$ if $j \neq i$. This leads to a diagonal score matrix

$$\begin{pmatrix} v_1(x_1^*) & 0 & \cdots & 0 \\ 0 & v_2(x_2^*) & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & v_n(x_n^*) \end{pmatrix}.$$

By construction, the overall value of consequence x^{i*} is the weight of the i-th attribute ($v_A(x^{i*}) = v_i(x_i^{i*}) = v_i(x_i^*) = w_i$). It then follows that paired statements (I, J) about the importance of attributes can be captured through the corresponding constraints (6)–(8), stated in terms of scores $v_A(x^{i*}) = w_i, i = 1, \ldots, n$. A benefit of this approach is that the resulting attribute weights are properly linked to ranges between the least and most preferred achievement levels under all attributes.

The linear inequality formulation in Theorem 2 makes it possible to synthesize incomplete ordinal preference information that is stated within *different* subsets of alternatives $X'^1, \ldots, X'^k \subseteq X$, but with regard to the *same* attribute set A'. Specifically, assume that the DM provides preference statements for the alternatives in sets $X'^l, l = 1, \ldots, k$ through paired combinations $(I_1'^l, J_1'^l), \ldots, (I_{n_l}'^l, J_{n_l}'^l)$ such that $|I_i'^l| \geq |J_i'^l|, i = 1, \ldots, n_l$ and $J_i'^l$ is sequential. Then all these statements can be captured by appending the constraints of Theorem 2 so that the variables $z_j^l, y_j^l(x^i)$ are made contingent on the particular set of alternatives X'^l (i.e, $x^i \in X'^l, j \in \cup_{i=1}^{n_l}\{\min J_i'^l - 1, \max J_i'^l\} \cap \{1, \ldots, |X'^l|\}$).

4 An Example

We illustrate the use of RICHER with an example where the CEO of a medium-sized company is about to re-locate the company's office. There are eight attributes: a_1: *size* of the office, a_2: *rental costs*, a_3: *renovation need*, a_4: *car park* opportunities, a_5: *means of communication* (i.e., public transport), a_6: *distance to city center*, a_7: *other facilities* (e.g., restaurants etc.), and a_8: *habitability* of the office and neighborhood.

Insert Table 1 about here

There are 12 alternatives (see Table 1). Under attributes a_1, a_2, a_4 and a_6, the CEO's preferences for the alternatives are captured through score intervals in the normalized $[0, 1]$-range. For attribute a_2, for instance, the value intervals are obtained through linear interpolation (i.e. monthly rent c, $1800 \leq c \leq 3200$, yields a score of $[3200 - c]/1400$). For attributes a_4 and a_6, the score intervals are generated so that if there is a strict preference for one alternative over another, the intervals do not overlap.

Insert Table 2 about here

The alternatives are evaluated by using categorized performance levels with regard to the four other attributes a_3, a_5, a_7 and a_8. In effect, such levels are a means of capturing incomplete ordinal preference information through paired sets of alternatives I and rankings J (see Table 3). For instance, alternative x^2 is ranked as the best one with regard to attribute a_3 (renovation need), while alternatives $x^1, x^5, x^6, x^7, x^{11}$ are among the five least preferred alternatives with regard to this attribute, because they may involve a considerable renovation need. The set of possible rankings for alternative x^7 is not sequential, because this alternative may require either considerable or small renovation need (but not intermediate): it is therefore associated with the rankings $\{3, 4, 8, 9, 10, 11, 12\}$.

Insert Table 3 about here

The conversion of paired sets I, J into linear constraints is illustrated by two examples. First, the above preference statement based on $I_{\{a_3\},X} = \{x^7\}$, $J_{\{a_3\},X} = \{3, 4, 8, 9, 10, 11, 12\}$, can be dealt with by forming the complement sets $\overline{I} = \{x^1, x^2, \ldots, x^6, x^8, x^9, \ldots, x^{12}\}$ and $\overline{J} = \{1, 2, 5, 6, 7\}$, of which the latter can be divided into its maximal sequential sets $J_1 = \{1, 2\}$ and $J_2 = \{5, 6, 7\}$. The indexes j needed for developing the constraint system of Theorem 3 are thus $J_{ind} = \cup_{i=1}^{2} \{\min J_i - 1, \max J_i\} \cap \{1, \ldots, 11\} =$

$\{2,4,7\}$. For the pairs $\overline{I}, J_1$ and $\overline{I}, J_2$, the constraints (6)–(7)

$$\begin{aligned} z_2 &\leq v(x^i) + (1 - y_2(x^i))M \\ v(x^i) &\leq z_2 + y_2(x^i)M, \\ z_4 &\leq v(x^i) + (1 - y_4(x^i))M, \\ v(x^i) &\leq z_4 + y_4(x^i)M, \\ z_7 &\leq v(x^i) + (1 - y_7(x^i))M, \\ v(x^i) &\leq z_7 + y_7(x^i)M \end{aligned}$$

must hold for all $x^i \in X$. Constraints (8) yield

$$\sum_{x^i \in X} y_2(x^i) = 2, \qquad \sum_{x^i \in X} y_4(x^i) = 4, \qquad \sum_{x^i \in X} y_7(x^i) = 7$$

while applying the constraint (10) for J_1 and J_2 gives

$$\sum_{x^i \in \overline{I}} y_2(x^i) = 2, \qquad \sum_{x^i \in \overline{I}} \left[y_7(x^i) - y_4(x^i)\right] = 3.$$

The optional monotonicity constraints (12) are $y_2(x^i) \leq y_4(x^i) \leq y_7(x^i), \forall\, x^i \in X$.

Second, in the paired set $I_{\{a_3\},X} = \{x^3, x^8, x^{12}\}$, $J_{\{a_3\},X} = \{4,5,6,7\}$, the maximal sequential subsets of $\overline{J}$ are $\{1,2,3\}$ and $\{8,9,10,11,12\}$. The set J_{ind} therefore contains indexes $\{2,3,4,7\}$, which makes it necessary to add the constraints (6)–(8) for $j = 3$ and $x^i \in X$:

$$\begin{aligned} z_3 &\leq v(x^i) + (1 - y_3(x^i))M, \\ v(x^i) &\leq z_3 + y_3(x^i)M, \\ \sum_{x^i \in X} y_3(x^i) &= 3. \end{aligned}$$

Developing the constraint (10) for the ranking sets $\{1,2,3\}$ and $\{8,9,10,11,12\}$ gives equalities $\sum_{x^i \in \overline{I}} y_3(x^i) = 3$ and $\sum_{x^i \in \overline{I}} [1 - y_7(x^i)] = 5$, while the new monotonicity constraints are $y_2(x^i) \leq y_3(x^i) \leq y_4(x^i)\ \forall\, x^i \in X$.

While the above statements were made with regard to the singleton attribute set $A' = \{a_3\}$, other attributes (and subsets of attributes) can be dealt with in the same way. In writing the corresponding constraints, it is useful to indicate the set of relevant attributes $y_j^{A'}(x^i), z_j^{A'}$, to highlight that the binary variables are contingent on what attributes are being considered.

Apart from the information in Tables 2 and 3, the CEO provides a full rank-ordering for the relative importance of attributes $r(a_1, \ldots, a_8) = (1, 2, \ldots, 8)$ (which is equivalent to $w_1 \geq w_2 \geq \ldots \geq w_8$)

and assigns a weight of $w_1 = 0.50$ to the most important attribute, office size. A lower bound of $1/[3n] = 1/24 \approx 0.0417$ is imposed on all weights to ensure that all attributes receive sufficient weight (cf. Salo and Punkka 2005).

In addition, the CEO provides two holistic statements, i.e., (i) alternative x^4 is preferred to x^1 and (ii) alternative x^1 is preferred to x^3. Adding the corresponding constraints $\mu_A(x^4, x^1) \geq 0$, and $\mu_A(x^1, x^3) \geq 0$ to the preference model now leads to the pairwise bounds $\mu_A(x^j, x^i)$ in Table 4 where non-negative pairwise bounds are written in boldface to indicate the presence of dominance.

Insert Table 4 about here

In Table 4, there are five non-dominated alternatives $x^5, x^7, x^8, x^9, x^{10}$. Because all four decision rules in Table 5 provide support for alternative x^5, this alternative could be presented to the DM as the recommended choice.

The required computations in this example were not time-consuming: for instance, the determination of pairwise bounds and overall value intervals for all 12 alternatives involved $12 \times 11 + 12 \times 2 = 156$ MILPs which were solved in 14 seconds on a Pentium III at 800 MHz with 256 MB RAM.

Insert Table 5 about here

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5 Conclusion

In response to the difficulties of eliciting complete preference information, we have presented the RICHER method which allows the DM to submit incomplete ordinal information about the relative importance of attributes, as well as the relative performance of alternatives with regard to these attributes. This information is elicited through paired sets of alternatives and associated rankings (e.g., 'alternatives 1 and 2 are among the three most preferred alternatives with regard to cost', or 'alternative 1 is not the most preferred one with regard to environmental factors'). In effect, such statements extend the use of incomplete ordinal information from the context of attribute weighting (Salo and Punkka 2005) to the comparison of alternatives. The resulting extension is also computationally attractive, because the MILP formulations presented in this paper make it possible to combine incomplete ordinal preference information with any other preference statements that correspond to linear constraints on the model parameters.

The elicitation of incomplete ordinal information may be particularly useful when the number of alternatives is large, because such information helps eliminate dominated alternatives so that the remaining elicitation efforts can be focused on the non-dominated ones. This kind of *screening* may be possible, for example, when the available data is incomplete for some alternatives but the DM is, nevertheless, capable of assigning preliminary rankings to the alternatives; or when the alternatives are evaluated with regard to intangible attributes for which ordinal scales may be more suitable than cardinal measurements.

This research suggests several avenues for further work. For example, behaviorally oriented empirical studies on the comparison of different elicitation procedures are needed, in order to develop guidelines that assist in the establishment of dominance relations. There is also a need for related decision support tools. Towards this end, we are in the process of developing a decision support system *RICHER Decisions*©, which will assist the execution of such studies as well as real-life applications.

Appendix

Proof of Lemma 2: Let $v(x^i), z_j, y_j(x^i),\ x^i \in X', j = 1, \ldots, m'-1$ be a feasible solution to (6)–(8). Define sets $I^+(j) = \{i \mid y_j(x^i) = 1\}, j = 1, \ldots, m'-1$. Constraint (8) implies $|I^+(j+1)| = |I^+(j)| + 1$ so that we may choose i^* such that $i^* \in I^+(j+1), i^* \notin I^+(j)$. For this i^*, constraints (6)–(7) give $z_{j+1} \leq v(x^{i^*})$ and $v(x^{i^*}) \leq z_j$, as required. The equality $|I^+(1)| = 1$ implies that there exists some $x^{i_1} \in X'$ such that $1 \geq v(x^{i_1}) \geq z_1$. Similarly, $|I^+(m'-1)| = m'-1$ means that there exists $x^{i_{m'-1}}$ such that $0 \leq v(x^{i_{m'-1}}) \leq z_{m'-1}$.

□

Proof of Theorem 1: $\subseteq$: Take any $v \in S(I,J)$. Then $\exists\ r \in R(I,J)$ such that $v \in S(r)$, i.e., $v(x^i) \geq v(x^j)$ whenever $r(x^i) < r(x^j)$. Define index numbers $i_1, \ldots, i_{m'}$ so that $r(x^{i_k}) = k$. Based on v, let $z_j = v(x^{i_j})$ and $y_j(x^{i_k}) = 1$ if $k \leq j$ and $y_j(x^{i_k}) = 0$ if $k > j$. We show that $v, z_j, y_j(x^{i_k})$ satisfy constraints (6)–(9).

Take any $j, k \in \{1, \dots, m'\}$. If $k \leq j$, then $y_j(x^{i_k}) = 1$ which can be inserted into (6) to obtain $z_j \geq v(x^{i_k}) + (1 - y_j(x^{i_k}))M = v(x^{i_k})$. If $j < k$, then $y_j(x^{i_k}) = 0$ and $v(x^{i_k}) \leq z_j$ so that (7) is satisfied. Due to the M-coefficient, other possibilities in (6)–(7) hold, too. Constraint (8) holds, because $\sum_{l=1}^{m'} y_j(x^{i_l}) = \sum_{l=1}^{j} \underbrace{y_j(x^{i_l})}_{=1} + \sum_{l=j+1}^{m'} \underbrace{y_j(x^{i_l})}_{=0} = j$.

Take any $j \in J$. Because $|I| \geq |J|$, there exists $x^{i_j} \in I$ such that $r(x^{i_j}) = j$. For this x^{i_j}, we have $y_j(x^{i_j}) = 1, y_{j-1}(x^{i_j}) = 0$ so that there is a positive term on the left side of (9). The definition of $y_j(x^i)$ gives $y_j(x^i) \geq y_j(x^{i-1})$, meaning that the sum in (9) is strictly positive. It is also integer-valued and bounded from above by one, because the use of (8) gives $\sum_{x^i \in I}[y_j(x^i) - y_{j-1}(x^i)] \leq \sum_{x^i \in X'}[y_j(x^i) - y_{j-1}(x^i)] = j - (j-1) = 1$. Thus, all constraints are satisfied.

$\supseteq$: Let $v(x^i), z_j, y_j(x^i)$ be a feasible solution to the constraints (6)–(9). We show that (i) there exists a rank-ordering r which is compatible with the sets I, J and (ii) that $v(x^i) \geq v(x^j)$ if $r(x^i) < r(x^j)$.

Based on the solution to (6)–(9), define a sequence of alternatives $x^{i_1}, \dots, x^{i'_m}$ so that x^{i_1} is the unique alternative such that $y_1(x^{i_1}) = 1$. For $j > 1$, choose $x^{i_j} \in X' \setminus \{x^{i_1}, \dots, x^{i_{j-1}}\}$ such that $y_j(x^{i_j}) = 1$; such an alternative exists due to constraint (8). If there is more than one possible candidate for choosing x^{i_j}, this same constraint implies that there are $x'', x' \neq x^{i_j}$ such that $y_{j-1}(x'') = 1, y_j(x'') = 0, y_{j-1}(x') = 0, y_j(x') = 1$. Inserting these parameters into (6)–(7) gives

$$\begin{aligned} v(x^{i_j}) &\leq z_{j-1}, & z_j &\leq v(x^{i_j}) \\ z_{j-1} &\leq v(x''), & v(x'') &\geq z_j \\ v(x') &\leq z_{j-1}, & z_j &\leq v(x') \end{aligned}$$

which imply $v(x^{i_j}) = v(x'') = v(x')$. Now, define binary variables $y'_j(x^i)$ which are the same as $y_j(x^i)$ for all $j = 1, \dots, m', x^i \in X'$ except for $y'_j(x'') = 1, y'_j(x') = 0$. Due to the value equality $v(x^{i_j}) = v(x'') = v(x')$, $v(x^i), z_j, y'_j(x^i)$ is also a feasible solution to (6)–(9). This process generates the sequence $x^{i_1}, \dots, x^{i'_m}$ and corresponding binary variables $y'_j(x^i)$'s with the property $y'_{j-1}(x^i) = 0$ whenever $y'_j(x^i) = 0$. Define a rank-ordering r by setting $r(x^{i_j}) = j$; by construction, the ranking of x^i is the least j such that $y'_j(x^i) = 1$.

Take any $j \in J$. Then constraint (9) gives $\sum_{x^i \in I}[y'_j(x^i) - y'_{j-1}(x^i)] = 1$, meaning for some $x^i \in I$, $y_j(x^i) = 1, y_{j-1}(x^i) = 0$. But this implies that $r(x^i) = j$, i.e., $r \in R(I, J)$.

To complete the proof, we show that $(v(x^1), \dots, v(x^{m'})) \in S(r)$. If $r(x^{i_k}) < r(x^{i_l}), i_k \neq i_l$, there exists j_k, j_l such that $y'_{j_k}(x^{i_k}) = 1, y'_{j_k-1}(x^{i_k}) = 0, y'_{j_l}(x^{i_l}) = 1, y'_{j_l-1}(x^{i_l}) = 0$. The definition of r implies $j_k < j_l$ so that $y'_{j_l-1}(x^{i_k}) = 1$. For $y_{j_l-1}(x^{i_k}) = 1$ and $y_{j_l-1}(x^{i_l}) = 0$, constraints (6)–(7) give

$$\begin{aligned} z_{j_l-1} &\leq v(x^{i_k}) \\ v(x^{i_l}) &\leq z_{j_l-1} \end{aligned}$$

so that $v(x^{i_k}) \geq v(x^{i_l})$, as required.

$\square$

Proof of Lemma 3: Clearly $|I| \geq |J_i| \; \forall i = 1, \ldots, k$. For any $i \in \{1, \ldots, k\}$, the rank-ordering r belongs to $R(I, J_i)$ iff $r^{-1}(j) \in I \; \forall j \in J_i$. Thus,

$$\begin{aligned}
\bigcap_{i=1}^{k} R(I, J_i) &= \bigcap_{i=1}^{k} \left\{ r \mid r^{-1}(j_i) \in I \; \forall j_i \in J_i \right\} \\
&= \left\{ r \mid r^{-1}(j_i) \in I \; \forall j_i \in J_i, i = 1, \ldots, k \right\} \\
&= \left\{ r \mid r^{-1}(j) \in I \; \forall j \in \cup_{i=1}^{k} J_i \right\} \\
&= \left\{ r \mid r^{-1}(j) \in I \; \forall j \in J \right\} = R(I, J).
\end{aligned}$$

$\square$

Proof of Theorem 2: Without loss of generality, we may assume that $1, m' \notin J$, as the cases $1, m' \in J$ are similar but more straightforward variants of the following proof.

$\supseteq$: Let $v(x^i), z_j, y_j(x^i)$ be a feasible solution to the constraints (6)–(8) and (10) for $j \in \{j^- - 1, j^+\} \cap \{1, \ldots, m'\}, x^i \in X'$. Now, partition X' into

$$\begin{aligned}
X^+ &= \left\{ x^i \in X' \mid y_{j^- - 1}(x^i) = 0, y_{j^+}(x^i) = 0 \right\}, \\
X^- &= \left\{ x^i \in X' \mid y_{j^- - 1}(x^i) = 1, y_{j^+}(x^i) = 1 \right\}, \\
X^p &= \left\{ x^i \in X' \mid y_{j^- - 1}(x^i) = 1, y_{j^+}(x^i) = 0 \right\}, \\
X^m &= \left\{ x^i \in X' \mid y_{j^- - 1}(x^i) = 0, y_{j^+}(x^i) = 1 \right\}.
\end{aligned}$$

First, if $X^p = \emptyset$, we have $y_{j^- - 1}(x^i) \leq y_{j^+}(x^i), \; \forall x^i \in X'$. Constraint (8) gives $|X^-| = j^- - 1$ for $j = j^- - 1$ and $|X^-| + |X^m| = j^+$ for $j = j^+$. Because these four sets form a partition of X' and X^p is empty, it follows that $|X^+| + |X^-| + |X^m| = m'$. Subtracting $|X^-| + |X^m| = j^+$ from this equality leads to $|X^+| = m' - j^+$. It thus follows that $|X^m| = m' - |X^+| - |X^-| = j^+ - j^- + 1 = |J|$.

Constraint (10) gives $\sum_{x^i \in I} \left[y_{j^+}(x^i) - y_{j^- - 1}(x^i) \right] = \sum_{x^i \in I \cap X^-} 0 + \sum_{x^i \in I \cap X^m} 1 + \sum_{x^i \in I \cap X^+} 0 = |J|$ so that $|I \cap X^m| = |J|$. But since $|X^m| = |J|$, this implies $X^m \subseteq I$.

Constraints (6)–(7) imply $v(x^-) \geq z_{j^- - 1} \geq v(x^m) \geq z_{j^+} \geq v(x^+)$ whenever $x^- \in X^-, x^m \in X^m, x^+ \in X^+$. Thus, there exists a sequence $i_1, \ldots, i_{m'}$ such that the values can be put in a sequence $v(x^{i_1}) \geq \ldots \geq v(x^{i_{m'}})$ where $x^{i_1}, \ldots, x^{i_{j^- - 1}} \in X^-, x^{i_{j^-}}, \ldots, x^{i_{j^+}} \in X^m, x^{i_{j^+ + 1}}, \ldots, x^{i_{m'}} \in X^+$. Setting

$r(x^{i_k}) = k$ defines a rank-ordering where the rankings in J are attained by alternatives in I and where $v(x^{i_k}) \geq v(x^{i_l})$ whenever $r(x^{i_k}) < r(x^{i_l})$. Thus $v \in S(I, J)$.

Second, if $X^p \neq \emptyset$, constraints (6)–(7) imply $z_{j^- -1} \leq v(x^i) \leq z_{j^+}$ for any $x^i \in X^p$, which leads to the inequality $z_{j^- -1} \leq z_{j^+}$. This inequality cannot be strict; for if $z_{j^- -1} < z_{j^+}$, then noting that $z_{j^+} \leq v(x^i) \leq z_{j^- -1}, x^i \in X^m$ would imply that X^m is empty; this, in turn, would lead to a contradiction, because $\sum_{x^i \in I} \left[y_{j^+}(x^i) - y_{j^- -1}(x^i)\right] = \sum_{x^i \in I \cap (X^+ \cup X^-)} 0 + \sum_{x^i \in I \cap X^p} (-1) \leq 0 < |J|$, in violation of (10). Thus, if $X^p \neq \emptyset$, then $z_{j^- -1} = z_{j^+} := z$. Constraints (6)–(7) imply that $v(x^-) \geq z = v(x^{m,p}) \geq v(x^+)$ for $x^- \in X^-, x^{m,p} \in X^m \cup X^p, x^+ \in X^+$.

By (10), $\sum_{x^i \in I} \left[y_{j^+}(x^i) - y_{j^- -1}(x^i)\right] = \sum_{x^i \in I \cap X^p} (-1) + \sum_{x^i \in I \cap X^m} 1 = |I \cap X^m| - |I \cap X^p| = |J|$, and hence $|(I \cap X^p) \cup (I \cap X^m)| \geq |J|$. There therefore exists a set $I' \subseteq (I \cap X^p) \cup (I \cap X^m)$ such that $|I'| = |J|$.

Because $|X^-| + |X^p| = j^- - 1$ by (8), it follows that $|X^-| < j^- - 1$. For $j = j^+$, constraint (8) gives $|X^-| + |X^m| = j^+$. Since the four sets form a partition of X', we have $|X^+| + |X^-| + |X^p| + |X^m| = m'$. Subtracting the former of these equalities from the latter gives $|X^+| + |X^p| = m' - j^+$ so that $|X^+| < m' - j^+$.

Now, let $i_1, \ldots, i_{m'}$ be a sequence such that the values are ordered as $v(x^{i_1}) \geq \ldots \geq v(x^{i_{m'}})$ where $x^{i_1}, \ldots, x^{i_{|X^-|}} \in X^-$, $x^{i_{|X^-|+1}}, \ldots, x^{i_{m'-|X^+|}} \in X^p \cup X^m$, $x^{i_{m'-|X^+|+1}}, \ldots, x^{i_{m'}} \in X^+$ and where the j-th position for any $j \in J$ obtained by $x^{i_j} \in I'$. Such a sequence exists, since $v(x^{i_k}) = v(x^{i_l})$ for any $k, l \in X^m \cup X^p$; it also defines a rank-ordering such that the rankings in J are attained by alternatives in I and $v(x^{i_k}) \geq v(x^{i_l})$ whenever $r(x^{i_k}) < r(x^{i_l})$, i.e., $v \in S(I, J)$.

$\subseteq$: Let $(v(x^1), \ldots, v(x^m)) \in S(I, J)$. Then, by Theorem (1), there exist $z_j, y_j(x^i)$ such that constraints (6)–(8) are satisfied for $x' \in X', j = 1, \ldots, m' - 1$; thus they hold for $j^- - 1, j^+$ whenever these are in the range $\{1, \ldots, m' - 1\}$. Adding equations (9) for $j = j^-, \ldots, j^+$ shows that (10) holds.

$\square$

Proof of Theorem 3: $\subseteq$: If $s \in S^\cap$, there is a solution $v(x^i), z_j, y_j(x^i)$ such that, for any $l \in \{1, \ldots, k\}$, constraints (6)–(8) hold for $x^i \in X', j \in \{\min J_l - 1, \max J_l\} \cap \{1, \ldots, m' - 1\}$ and (10) holds for J_l. Thus, $s \in S^2(I_l, J_l)$, whereby $s \in S(I_l, J_l)$ by Theorem 2. Because this holds for all $l \in \{1, \ldots, k\}$, it follows that $s \in \bigcap_{l=1}^k S(I_l, J_l)$, as required.

$\supseteq$: First, assume that $v \in S(I_1, \ldots, I_k; J_1, \ldots, J_l) \cap S^\circ_{A'}$, where $S^\circ_{A'} = \{s \in S_0 \mid x^k \neq x^l \Rightarrow v_{A'}(x^k) \neq v_{A'}(x^l)\}$. Then alternatives $x^i \in X'$ can be ordered so that $v(x^{i_1}) > \ldots > v(x^{i_{m'}})$. Let $z_j = v(x^{i_j})$ and

$y_j(x^{i_k}) = 1$ if $k \geq j$ and $y_j(x^{i_k}) = 0$ if $k < j$. By construction, constraints (6)–(8) hold for $z_j, v(x^{i^i}), y_j(x^i)$ and constraint (10) holds for $J_l, l = 1, \ldots, k$. Thus, $v \in S^{\cap}$.

Second, assume that $s \in S(I_1, \ldots, I_k; J_1, \ldots, J_l) \setminus S^{\circ}_{A'}$. Then X' can be partitioned into subsets $X'_1, \ldots, X'_q$ such that any distinct alternatives x^i, x^j have the same value if and only if they belong to same subset. Thus, alternatives $x^i \in X'$ can be ordered so that $v(x^{i_1}) \geq \ldots \geq v(x^{i_{m'}})$ where the j-th inequality is strict iff x^j, x^{j+1} are in different subsets of the partition of X'. Based on this ordering, the solution $z_j, y_j(x^{i_k})$ can now be defined as in the first case. It is straightforward to show that all the constraints are fulfilled, proving $s \in S^{\cap}$.

$\square$

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Table 1: Office alternatives

	a_1	a_2	a_3	a_4		a_5	a_6	a_7	a_8
		Rent		Car		Public	Distance		
Alt.	Area	(euros)	Renovation need	park	Garage	transport	to center	Other facilities	Habitability
x^1	180 m^2	2000	considerable	13	no	quite bad	12 km	intermediate	great
x^2	240 m^2	3000	no	13	no	good	15 km	good	bad
x^3	210 m^2	2800	intermediate	2	no	great	0 km	great	good
x^4	214 m^2	2000	very small	13	yes	bad	25 km	intermediate	good
x^5	300 m^2	3200	considerable	13	yes	good	4 km	good or great	very good
x^6	170 m^2	1800	considerable	5	no	good	0 km	great	good
x^7	250 m^2	2600	small / considerable	13	yes	intermediate	7 km	good	intermediate
x^8	260 m^2	2650	intermediate	10	no	good	10 km	intermediate	intermediate
x^9	262 m^2	2400	big	13	yes	good	10 km	intermediate	very good
x^{10}	241 m^2	2500	small	11	yes	intermediate	7 km	good	good
x^{11}	198 m^2	2200	considerable	13	no	bad	17 km	good	good
x^{12}	201 m^2	2000	intermediate	7	no	bad	22 km	quite bad	intermediate

Table 2: Normalized scores

x^j	$v_1^N(x_1^j)$	$v_2^N(x_2^j)$	$v_4^N(x_4^j)$	$v_6^N(x_6^j)$
x^1	0.13	0.86	$[0.90, 0.95]$	$[0.27, 0.40]$
x^2	0.66	0.14	$[0.90, 0.95]$	$[0.17, 0.30]$
x^3	0.43	0.29	$[0.20, 0.30]$	$[1.00, 1.00]$
x^4	0.47	0.86	$[1.00, 1.00]$	$[0.00, 0.00]$
x^5	1.00	0.00	$[1.00, 1.00]$	$[0.70, 0.80]$
x^6	0.00	1.00	$[0.40, 0.50]$	$[1.00, 1.00]$
x^7	0.73	0.43	$[1.00, 1.00]$	$[0.50, 0.65]$
x^8	0.79	0.39	$[0.80, 0.88]$	$[0.30, 0.45]$
x^9	0.80	0.57	$[1.00, 1.00]$	$[0.30, 0.45]$
x^{10}	0.67	0.50	$[0.95, 0.98]$	$[0.50, 0.65]$
x^{11}	0.32	0.71	$[0.90, 0.95]$	$[0.10, 0.20]$
x^{12}	0.35	0.86	$[0.55, 0.65]$	$[0.01, 0.05]$

Table 3: Incomplete ordinal preference statements

Attribute	I	J
a_3	$\{x^2\}$	$\{1\}$
	$\{x^4\}$	$\{2\}$
	$\{x^{10}\}$	$\{3,4\}$
	$\{x^7\}$	$\{3,4,8,9,10,11,12\}$
	$\{x^3,x^8,x^{12}\}$	$\{4,5,6,7\}$
	$\{x^9\}$	$\{7,8\}$
	$\{x^1,x^5,x^6,x^{11}\}$	{8,9,10,11,12 }
a_5	$\{x^3\}$	$\{1\}$
	$\{x^2,x^5,x^6,x^8,x^9\}$	$\{2,3,4,5,6\}$
	$\{x^7,x^{10}\}$	$\{7,8\}$
	$\{x^1\}$	$\{9\}$
	$\{x^4,x^{11},x^{12}\}$	$\{10,11,12\}$
a_7	$\{x^3,x^6\}$	$\{1,2,3\}$
	$\{x^5\}$	$\{1,2,3,4,5,6,7\}$
	$\{x^2,x^7,x^{10},x^{11}\}$	$\{3,4,5,6,7\}$
	$\{x^1,x^4,x^8,x^9\}$	$\{8,9,10,11\}$
	$\{x^{12}\}$	$\{12\}$
a_8	$\{x^1\}$	$\{1\}$
	$\{x^5,x^9\}$	$\{2,3\}$
	$\{x^3,x^4,x^6,x^{10},x^{11}\}$	$\{4,5,6,7,8\}$
	$\{x^7,x^8,x^{12}\}$	$\{9,10,11\}$
	$\{x^2\}$	$\{12\}$

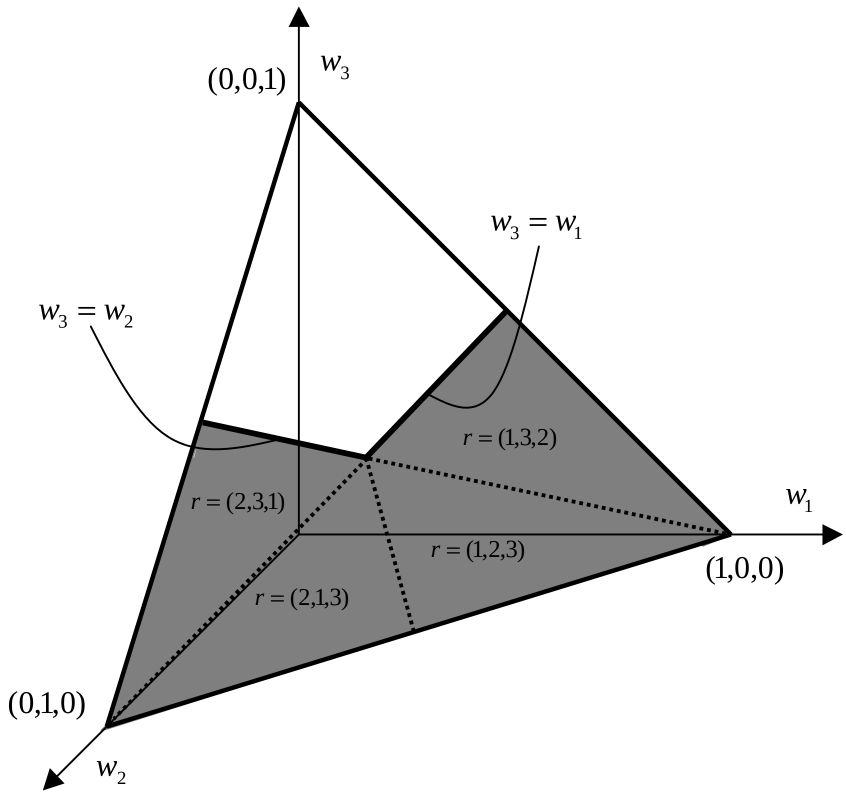
Table 4: Pairwise bounds

	Alternative											
	x^1	x^2	x^3	x^4	x^5	x^6	x^7	x^8	x^9	x^{10}	x^{11}	x^{12}
$\mu(x^1, x^i)$	–	-0.263	**0.000**	-0.282	-0.428	-0.058	-0.350	-0.354	-0.323	-0.368	-0.089	-0.153
$\mu(x^2, x^i)$	**0.033**	–	**0.033**	-0.123	-0.312	-0.025	-0.171	-0.178	-0.274	-0.199	-0.056	-0.048
$\mu(x^3, x^i)$	-0.040	-0.263	–	-0.282	-0.428	-0.058	-0.350	-0.354	-0.323	-0.368	-0.089	-0.153
$\mu(x^4, x^i)$	**0.000**	-0.244	**0.000**	–	-0.376	**0.007**	-0.229	-0.236	-0.314	-0.264	-0.071	**0.004**
$\mu(x^5, x^i)$	**0.143**	**0.049**	**0.143**	**0.029**	–	**0.137**	-0.051	-0.061	-0.155	-0.038	0.105	**0.102**
$\mu(x^6, x^i)$	-0.238	-0.390	-0.238	-0.397	-0.655	–	-0.464	-0.481	-0.561	-0.482	-0.308	-0.263
$\mu(x^7, x^i)$	**0.076**	-0.043	**0.076**	-0.079	-0.309	**0.058**	–	-0.137	-0.217	-0.150	**0.016**	**0.035**
$\mu(x^8, x^i)$	**0.119**	-0.089	**0.119**	-0.115	-0.215	**0.103**	-0.177	–	-0.188	-0.192	**0.067**	**0.039**
$\mu(x^9, x^i)$	**0.172**	-0.042	**0.172**	-0.028	-0.173	**0.203**	-0.129	-0.135	–	-0.106	**0.167**	**0.087**
$\mu(x^{10}, x^i)$	**0.143**	-0.070	**0.143**	-0.099	-0.239	**0.085**	-0.158	-0.067	-0.164	–	**0.054**	**0.102**
$\mu(x^{11}, x^i)$	-0.152	-0.298	-0.152	-0.310	-0.569	-0.150	-0.382	-0.388	-0.465	-0.399	–	-0.183
$\mu(x^{12}, x^i)$	-0.115	-0.349	-0.115	-0.361	-0.480	-0.079	-0.433	-0.439	-0.418	-0.452	-0.177	–

Table 5: Decision rules

Decision rule	Objective function	x^5	x^7	x^8	x^9	x^{10}
Maximax	$\max V(x)$	**0.892**	0.758	0.762	0.834	0.768
Maximin	$\min V(x)$	**0.571**	0.505	0.511	0.556	0.485
Central values	$(\max V(x) + \min V(x))/2$	**0.732**	0.632	0.637	0.695	0.627
Minimax regret	$\max_{x^i \neq x}[-\mu(x, x^i)]$	**0.155**	0.309	0.215	0.173	0.239

Figure 1



Captions

Figure 1. Non-convex feasible region